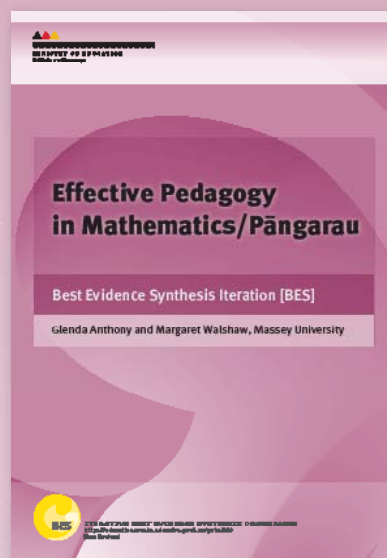# Support students to make their own choice of tools and representations when solving problems

This is one of a series of cases that illustrate the findings of the best evidence syntheses (BESs). Each is designed to support the professional learning of educators, leaders and policy makers.

Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

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BES cases: Insight into what works

The best evidence syntheses (BESs) bring together research evidence about 'what works' for diverse (all) learners in education. Recent BESs each include a number of cases that describe actual examples of professional practice and then analyse the findings. These cases support educators to grasp the big ideas behind effective practice at the same time as they provide vivid insight into their application.

Building as they do on the work of researchers and educators, the cases are trustworthy resources for professional learning.

Using the BES cases

The BES cases overview provides a brief introduction to each of the cases. It is designed to help you quickly decide which case or cases could be helpful in terms of your particular improvement priorities.

Use the cases with colleagues as catalysts for reflecting on your own professional practice and as starting points for delving into other sources of information, including related sections of the BESs. To request copies of the source studies, use the Research Behind the BES link on the BES website.

The conditions for effective professional learning are described in the Teacher Professional Learning and development BES and condensed into the ten principles found in the associated International Academy of Education summary (Timperley, 2008).

Note that, for the purpose of this series, the cases have been re-titled to more accurately signal their potential usefulness.

Responsiveness to diverse (all) learners

Use the BES cases and the appropriate curriculum documents to design a response that will improve student outcomes
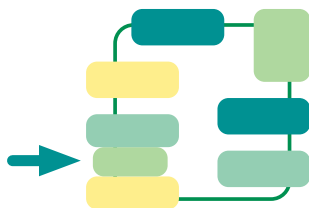
The different BESs consistently find that any educational improvement initiative needs to be responsive to the diverse learners in the specific context. Use the inquiry and knowledge-building cycle tool to design a collaborative approach to improvement that is genuinely responsive to your learners

Support students to make their own choice of tools and representations when solving problems

This case illustrates how cognitive conflict can be a resource for learning. When an apparent contradiction surfaced during problem solving, instead of circumventing discussion by providing an explanation, the teacher in the case ensured that all students were challenged to explore their own thinking, and encouraged to use a range of tools to solve the problem. In this way, the teacher allowed the students to focus on justifying their methods, based on the logic of mathematics.

Using tools to support learners' mathematical thinking

As we have seen in the previous CASEs, children's exploration of fractions can usefully involve a range of representational tools including drawings and diagrams, symbols, and manipulatives such as paper rectangles. These representations assist learners to focus on certain key features of fractions.

As discussed in chapter 5, tools can also support students' strategic thinking and help make their solution strategies visible to others. In CASE 2, Empson (2003) noted that allowing children to choose their own tools and make their own representations to solve equal-sharing problems fosters "an interesting diversity of thinking, which can contribute to richer understanding of the mathematics of fractions" (p. 35). For example, when using part–whole representations, sixths can be drawn in three ways: by making halves and partitioning each into thirds, by making thirds and partitioning each into halves; and by making a guess about how big a sixth is and partitioning the pieces one by one. Each method supports different mathematical representations of equivalence. The first justifies the equivalence of ½ and ³⁄₆; the second justifies the equivalence of ⅓ and ²⁄₆; and the third lends itself to the idea that ⁶⁄₆ is equivalent to 1.

CASE 4 examines the use of a real context, sharing cakes, in which students move from having a model 'for' division to a model 'of' division. Division of fractions is readily acknowledged as the most complex of the arithmetical operations (Ma, 1999). Fraction division can be explained as an extension of whole-number division categorisations—measurement division, partitive division, and the inverse of a Cartesian product. Division as the determination of a unit rate and division as the inverse of multiplication are two further important fraction-division interpretations (Sinicrope, Mick & Kolb, 2002). Traditionally, children have been taught division as a rule-based procedure, with little attempt to ground this procedure in a meaningful context. The researchers in this and several other studies claim that building connections with informal knowledge and encouraging students to explore multiple explanations allows students to develop a robust understanding of rational numbers.

CASE 4: Dividing the Cake

(from Sarage, 1992)

Mathematics teaching for diverse learners:
- involves respectful exchange of ideas;
- provides opportunities for children to resolve cognitive conflict;
- provides for both planned and spontaneous/informal learning;
- utilises tools as learning supports;
- provides opportunities for students to problematise activities based in realistic contexts;
- builds on students' prior knowledge and experiences.

Targeted learning outcomes
Solving problems involving division with fractions.

Learning context
A class discussion involving the solution of the problem "Four block cakes are to be divided into portions of three-fifths of a cake.   How many portions are there?"

Student activity

Following exploration of the problem, the students were required to share their solution strategies with the class.

Vivian: I'll show my diagram [going to the board to demonstrate her solution]. See, here are the four cakes (fig. 7.5). And you can see that you get six portions out of them.
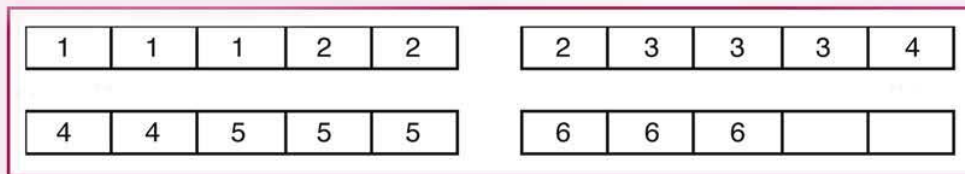
1	1	1	2	2		2	3	3	3	4
4	4	5	5	5		6	6	6		

Fig. 7.5. Four cakes divided into portions of $^3/_5$.

Claire: What about the left overs?

Vivian: [After a glance at her diagram] Okay, so it's 6 and $^2/_5$.

Jonathan: Wait a minute. When I do the calculation I get 6 and $^2/_3$.

Vivian: But look at the picture. You can see that it's $^2/_5$ left over.

Seeing the significance of the contradiction, the teacher interrupts their debate to make sure the rest of the class sees it too. Instructed to work with a partner, the students puzzle over what to do with the two pieces of cake that are left over after six people take their portions. When the class is called together again, Carole attempts her group's explanation:

Carole: The problem asks about portions. You can say that there are 6 portions with $^2/_5$ of a cake left over or you can say that there are 6 and $^2/_3$ portions.

Sandy: Look at something else in Vivian's diagram. She started with 4 cakes. Then she cut each cake into 5 pieces. So she had $4 \times 5 = 20$ pieces. Then she grouped those pieces by threes since 3 pieces make up a portion. So she got $20 \div 3 = 6^2/_3$ portions. She multiplied by the denominator and divided by the numerator. Like in flip and multiply!

Eleanor: I see something else in that diagram. You've got $^3/_5$ of a cake equal to a portion. But you can see that each cake is one portion plus another $^2/_3$ of a portion. That is, each cake is $^5/_3$ of a portion. So when you want to find out how many portions there are in 4 cakes, you can divide by the size of each portion ($4 \div ^3/_5$) or you can multiply by the number of portions per cake ($4 \times ^5/_3$). Amazing!

Both Sandy and Eleanor are describing the meaning they now find in the usually mysterious rule for dividing by fractions.

Learner outcomes

The classroom discourse of enquiry encouraged students' struggle to resolve conflicts or confusions in their thinking. The discussion resulted in students' productive reorganisation of the mathematical ideas into more complex levels of understanding. By allowing these students to proceed with an explanation, even when their initial answer was wrong, the teacher fostered an expectation that the mathematical authority resides within the mathematical justification, to be shared and endorsed by both teacher and students. Sharing the locus of authority meant that students in this class were free to develop confidence in their own methods and their own monitoring skills when deciding whether something made sense. Rather than trying to uncover what the teacher wanted, students were "free to focus their attention on developing justification for their methods and solutions based on the logic of mathematics" (Hiebert et al., 1997, p. 41).

- Students were able to create problems for themselves that were appropriate to their own levels of understanding.

These pedagogic strategies involve "not simply helping students to learn but, more fundamentally, learning from the learners" (p. 19).

References

Houssart, J. (2002). Simplification and repetition of mathematical tasks: a recipe for success or failure? *The Journal of Mathematical Behavior, 21*(2), 191-202.

Kieren, T., Davis, B., & Mason, R. (1996). Fraction flags: Learning from children to help children learn. *Mathematics Teaching in the Middle School, 2*(1), 14-19.

Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Turner, J., Meyer, D., Cox, K., Logan, C., DiCintio, M., & Thomas, C. (1998). Creating contexts for involvement in mathematics. *Journal of Educational Psychology, 90*(4), 730-745.