



MINISTRY OF EDUCATION

Te Tāhuhu o te Mātauranga

Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

Glenda Anthony and Margaret Walshaw, Massey University



ITERATIVE BEST EVIDENCE SYNTHESIS PROGRAMME
<http://educationcounts.edcentre.govt.nz/goto/BES>
New Zealand

Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

Glenda Anthony, Massey University
Margaret Walshaw, Massey University

New Zealand Ministry of Education

This report is one of a series of best evidence synthesis iterations (BESs) commissioned by the Ministry of Education. The Iterative Best Evidence Synthesis Programme is seeking to support collaborative knowledge building and use across policy, research and practice in education. BES draws together bodies of research evidence to explain what works and why to improve education outcomes, and to make a bigger difference for the education of all our children and young people.

Each BES is part of an iterative process that anticipates future research and development informing educational practice. This BES follows on from other BESs focused on quality teaching for diverse learners in early childhood education and schools. Its use will be informed by other BESs, focused on teacher professional learning and development and educational leadership. These documents will progressively become available at: <http://educationcounts.edcentre.govt.nz/goto/BES>

Feedback is welcome at best.evidence@minedu.govt.nz

Note: the references printed in purple refer to a list of URLs in Appendix 2. These are a selection of potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration.

Published by the Ministry of Education, Box 1666, Wellington, New Zealand 6140.
www.minedu.govt.nz

Copyright © Crown 2007

All rights reserved. Enquiries should be made to the publisher.

Glenda Anthony and Margaret Walshaw assert their moral right to be recognised as the authors of this work.

Dewey number 510.7

ISBN 978 0 7903 2624 5

Item number 32624

Contents

About the writers.....	vi
Advisory Group	vi
Ministry of Education advisory team	vi
External quality assurance	vi
Acknowledgments.....	vii
Forewords	viii
International	viii
Early Childhood Education	xi
NZEI Te Riu Roa	xii
Post Primary Teachers' Association	xiii
Teacher Educators	xiii
Māori-medium Mathematics.....	xiv
Pasifika.....	xiv
Chief Education Adviser, BES	xv
Authors' Preface.....	xviii
What is a Best Evidence Synthesis in Mathematics?.....	xviii
The importance of dialogue.....	xviii
Writing for multiple audiences	xviii
The BES as a catalyst for change	xix
Key features	xix
References.....	xx
Executive Summary	1
Key findings	1
Overall key findings	4
Gaps in the literature and directions for future research.....	4
1. Introduction.....	5
Mathematics in New Zealand.....	5
Pedagogical approaches and learner outcomes.....	5
Making a difference for all.....	6
Mathematical proficiency.....	7
Social, affective, and participatory outcomes.....	7
Diversity.....	8
Equity.....	8
Positioning the Best Evidence Synthesis.....	10
Overview of chapters	11
References.....	12
2. Framing the BES	14
Aims of the Best Evidence Synthesis	14
A comparative perspective	14
Effective numeracy teaching	16
Professional development and teacher knowledge	17
Equitable student access to learning.....	18
Conceptualising the synthesis	20
References.....	22
3. Early Years Mathematics Education	24
Introduction	24
Mathematical learning experiences and activities.....	25
Communities of practice	36

Centre-home links	41
Teacher knowledge and beliefs	45
Bridge to school.....	46
References	49
4. Mathematical Communities of Practice	54
Mathematics teaching for diverse learners demands an ethic of care.....	54
Mathematics teaching for diverse learners creates a space for the individual and the collective.....	61
Mathematics teaching for diverse learners demands explicit instruction	69
Mathematics teaching for diverse learners involves respectful exchange of ideas.....	72
Mathematics teaching for diverse learners demands teacher content knowledge, knowledge of mathematics pedagogy, and reflecting-in-action.....	81
References	87
5. Mathematical Tasks, Activities and Tools	94
Introduction	94
Tasks that are problematic and have a mathematical focus provide opportunities for mathematical thinking.....	94
Mathematics teaching for diverse learners ensures that tasks link to learners’ prior knowledge and experiences	109
Mathematical tasks need to provide opportunities for cognitive engagement press and for understanding	120
Mathematics teaching for diverse learners utilises tools as learning supports.....	126
Conclusion	140
References	141
6. Mathematics Practices outside the Classroom.....	151
The school	151
The role of the mathematics lead teacher or head of department in establishing productive teaching communities	154
Whole-school partnerships.....	157
School-home partnerships	160
Facilitating school, family, and community partnerships.....	162
Conclusion	171
References	172
7. A Fraction of the Answer.....	177
Introduction	177
Building on learners’ prior knowledge and experiences.....	178
Positioning participants as proficient learners.....	179
Challenging tasks with a mathematical focus	183
Using tools to support learners’ mathematical thinking.....	187
Task engagement and sense making	189
Respectful exchange of ideas	190
Belonging to a community of learners.....	195
Teacher knowledge: forms of ‘knowing’ fractions.....	197
Part-whole pedagogy	200
References	200
8. Endings Marking New Beginnings	202
Gaps and omissions	204
Final words	205

Reference.....	205
Appendix 1: Locating and Assembling BES Data.....	206
Selecting the evidence.....	207
Research sources in this BES report	208
Synthesising the data.....	209
References.....	210
Appendix 2: URLs of citations.....	211
Appendix 3: Glossary	213

About the writers

Glenda Anthony and Margaret Walshaw, both from the School of Curriculum and Pedagogy at Massey University, bring to this Best Evidence Synthesis (BES) decades of mathematics classroom teaching and educational research experience. They are acutely aware of the challenge that educators face in constructing a democratic mathematical community with which all students can identify. For them, making a positive difference to diverse learners' outcomes is a central educational issue. At the heart of their work is a concerted effort to illuminate how this issue is best addressed. In this synthesis, they report on the outcome of their deliberations over, and search for, what makes a difference for diverse learners in mathematics/pāngarau.

Advisory Group

A core Advisory Group membership was selected to provide expertise and critique in relation to the various focuses of the BES, including Māori and Pasifika learners, early childhood, primary and secondary sectors, and teacher education. The authors wish to thank the members of this group:

- Dr Ian Christensen (Massey University and He Kupenga Hao i te Reo)
- Dr Joanna Higgins (Victoria University of Wellington)
- Roberta Hunter (Massey University)
- Garry Nathan (Auckland University)
- Dr Sally Peters (Waikato University)
- Assoc. Prof. Jenny Young-Loveridge (Waikato University)

We also wish to acknowledge the supportive formative feedback received from Faith Martin (Director, Massey Child Care Centre), Brian Paewai (Runanga Kura Kaupapa Māori), Professor Anne Smith (University of Otago) and Johanna Wood (Principal, Queen Elizabeth College, Palmerston North).

Ministry of Education advisory team

The Ministry of Education, led by Dr Adrienne Alton-Lee, has guided the development of the synthesis. The team at the Ministry also gave us access to additional literature and demographic and trend data. We thank all of the team.

External quality assurance

Professor Paul Cobb from Vanderbilt University, US, has provided invaluable assistance. We would like to acknowledge his scholarly critique and thank him for his knowledgeable contribution to the synthesis.

Formative quality assurance was also provided by: Maggie Haynes (Unitec), Professor Derek Holton (University of Otago), Tamsin Meaney (EARU, University of Otago), Lynne Peterson, Tony Trinick (Auckland University), initial and ongoing Teacher Education (Victoria University of Wellington), the New Zealand Educational Institute and representation from the Post Primary Teachers' Association (Jill Gray). We wish to thank them all for their contributions.

Acknowledgments

The Ministry of Education extends special thanks to those who have contributed to different stages of this BES development through their participation in the BES Management Group. The advice and guidance from principal Diane Leggett, NZEI, and Judie Alison, Advisory Officer (Professional Issues), PPTA, have greatly strengthened this BES development. Particular thanks also to Robina Broughton and Linda Gendall, New Zealand Teachers Council, and to Ministry of Education colleagues.

The Chief Education Adviser acknowledges in particular the support and guidance provided by Malcolm Hyland and Ro Parsons through the partnership between BES and the Numeracy Development Project. The model of collaboration across research, practice and policy exemplified in that project has been an inspiration for the Iterative Best Evidence Synthesis Programme. Thanks to all those in the wider NDP community who have informed the BES development.

The Ministry of Education thanks Dr Fred Biddulph for his ongoing role in providing advice from the earliest formulation of the request for proposals through to a consideration of the final draft.

The Ministry of Education is indebted to Professor Bill Barton, Mathematics Education Unit, University of Auckland, for taking a proactive leadership role in bringing together teacher, teacher educator and research colleagues from across New Zealand to assist in scoping this BES at the outset.

Thanks for the deeply valued contribution made to the formative quality assurance and other advice offered to this BES development by Professor Paul Cobb, Vanderbilt University; Irene Cooper, Sandie Aikin, Cheryl Baillie and colleagues, NZEI, Jill Gray and Patrick McEntee, PPTA; Dr Mere Skerrett-White, Dr Maggie Haynes, Unitech, Dr Jo Higgins and colleagues at the Victoria University of Wellington College of Education; Dr Tamsin Meany, Professor Derek Holton, Lynne Petersen, Peter Hughes, Lynn Tozer and Michael Drake.

Thanks also for the valued participation of colleagues from initial teacher education institutions across New Zealand and Derek Glover, Secretary, New Zealand Association of Mathematics Teachers, in the formative quality assurance forum held for this BES development.

Thanks also for the significant contribution made to this and other BES developments through the advice given in the development of the Guidelines for Generating a Best Evidence Synthesis Iteration by the BES Standards Reference Group; The BES Māori Educational Research Advisory Group, the BES Pasifika Educational Research Advisory Group and Associate Professor Brian Haig, University of Canterbury.

Forewords

International

Even the casual visitor is struck by the dramatic changes that have occurred in New Zealand in the last 15 years. I have tuned in to local media on each of my four visits to get an initial sense of people's current concerns and issues. Based on this narrow sampling, the New Zealand of 1991 was an immensely likeable country that had seen better days and was struggling to find its place in a rapidly changing world. Although innovation and experimentation appeared to be the watchwords of the day, there seemed to be an undercurrent of apprehension and anxiety as people attempted to cope with economic disruption. Today, New Zealand continues to be an immensely likeable place, but the visitor immediately notices a quiet, understated self-assurance. It has become a largely prosperous country that, in a very real sense, has reinvented itself as a leading information economy in an increasingly globalised world. Refreshingly for the visitor from the United States, there appears to be widespread belief that government will approach problems pragmatically and is capable of solving them. If the Iterative Best Evidence Synthesis Programme is representative of New Zealand government in action, this belief would appear to be well founded.

Put quite simply, the Iterative BES Programme is the most ambitious effort I have encountered that uses rigorous scientific evidence to guide the ongoing improvement of an education system at a national level. The programme has a strong pragmatic bent and is clearly grounded in the hard-won experience of synthesising research findings to inform both policy and teachers' instructional practices. Four aspects of the programme are particularly noteworthy. The first is the overriding commitment to make the development of the best evidence syntheses transparent. This commitment takes concrete form in the exacting evaluation and feedback process that all BES reports undergo at each phase of their development, from the initial identification of relevant bodies of research literature through to the final critique and revision of the report. This is in the best traditions of science, where claims are justified in terms of the means by which they have been produced.

The second notable characteristic is a mature view of evidence and an emphasis on methodological and theoretical pluralism. This is important, given that attempts have been made in a number of countries, including the United States, to legislate what counts as scientific research in education on the basis of ideological adherence to a particular methodology. In taking an inclusive approach, the Iterative BES Programme acknowledges that different types of knowledge are of greatest use to teachers and to policymakers. Teachers make pedagogical decisions on the basis of a detailed understanding of specific students in particular classrooms at particular points in time. Policymakers, in contrast, typically need knowledge of trends and patterns that hold up across classrooms to make decisions that affect large numbers of students and teachers in multiple schools. Different methodologies are appropriate for developing these equally important types of knowledge.

The third noteworthy characteristic of the Iterative BES Programme is its focus on the explanatory power and coherence of theories. Priority is given to theories that give insight into learning processes and the specific means of supporting their realisation in classrooms. This pragmatic criterion is important in a field where theoretical perspectives continue to proliferate.

The final notable characteristic of the programme is its explicit attention to the issues of language and culture. This emphasis is clearly critical if New Zealand teachers and policymakers are to address the inequities inherent in the disturbingly large gaps in school achievement between children of different ethnic and racial groups. In keeping with the tenet of methodological and theoretical pluralism, the Iterative BES Programme uses group categories such as socioeconomic status, ethnicity, and culture as key variables in *assessing* efforts to achieve

equity. However, it avoids stereotyping children of particular racial, ethnic, or language groups by acknowledging the complexity of individual identity when *explaining* inequities in children's learning opportunities. Furthermore, the programme emphasises ecological models of learning that link what is happening in classrooms both to the institutional contexts in which classrooms are located and to issues of race, culture, and language. It is here that the full ambition of the programme becomes apparent: few viable models of this type currently exist in education. The BES writers are therefore charged with the task of synthesising in the true sense of the term, that is, to combine disparate and sometimes fragmented bodies of research into a single, unified whole. At the risk of understatement, this is a formidable challenge.

The writers of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau, Drs. Glenda Anthony and Margaret Walshaw, have risen to the challenge. They were charged with the daunting task of reviewing, organising, and synthesising all mathematics education research from the early childhood years through secondary school that relates classroom processes to student learning. On my reading, the resulting synthesis of over 600 research studies is directly relevant to teachers and will be educative for policymakers. The educative value of the report stems from Anthony and Walshaw's focus on what goes on in mathematics classrooms, thereby providing a window on the complexity of effective pedagogy. The forms of pedagogical practice that they identify as effective are ambitious because they involve high expectations for *all* children's mathematical learning. The goals at which these forms of pedagogy aim are best illustrated in chapter 7, A Fraction of the Answer, in which Anthony and Walshaw pull together the key insights of the preceding chapters as they present an integrated series of cases that focus on the learning and teaching of fractions. As this chapter makes clear, the instructional goals for fractions are not limited to ensuring that children can add, subtract, multiply, and divide fractions successfully. Instead, the instructional objectives also focus on children's development of a deep understanding of fractions as amounts or quantities. At an elementary level, children who are coming to understand fractions as quantities know that $\frac{1}{6}$ is smaller than $\frac{1}{5}$ because there will be more pieces when something is divided into 6 pieces than into 5 pieces, so the pieces must be smaller. At a more advanced level, students will be able to describe real world situations that involve multiplying and dividing fractional quantities. More generally, ambitious pedagogy focuses on central mathematical ideas and principles that give meaning to computational methods and strategies.

Anthony and Walshaw's review of the relevant research indicates that central mathematical ideas and principles cannot be directly transmitted to children. However, the research also shows that discovery approaches that place children in rich environments and simply encourage them to inquire are also ineffective. Effective pedagogy is complex because it requires teachers to achieve a significant mathematical agenda by taking children's current knowledge and interests as the starting point. As Anthony and Walshaw clarify, these forms of pedagogy involve a distinctive orientation towards teaching. First and foremost, the emphasis is on building on students' existing proficiencies rather than filling gaps in students' knowledge and remediating weaknesses. As a consequence, the teacher's focus when planning for instruction is not on students' limitations but on their current mathematical competencies and interests, as these constitute resources on which the teacher can build. More generally, effective mathematical pedagogy places students' reasoning at the center of instructional decision making. As a consequence, the ongoing assessment of students' reasoning is an integral aspect of instruction, not a separate activity conducted after the fact to check whether goals for students' learning have been achieved. A key characteristic of accomplished teachers is that they continually adjust instruction, as informed by these ongoing assessments.

One of the strengths of Anthony and Walshaw's synthesis is that it provides the reader with a concrete image of what effective mathematical pedagogy looks like. Anthony and Walshaw emphasise that a respectful, non-threatening classroom atmosphere in which all students feel comfortable in making contributions is necessary but not, by itself, sufficient. As they document, the research findings indicate unequivocally that it is also essential that classroom activity

and discourse focus explicitly on central mathematical ideas and processes. The selection of instructional tasks is therefore critical. On the one hand, it is important that task contexts or scenarios are accessible to all students, regardless of cultural background. On the other, the teacher should be able to capitalise on students' solutions to support their development of increasingly sophisticated forms of mathematical reasoning. Thus, when designing and selecting tasks, the teacher has to take account both of students' current competencies and interests and their long-term learning goals. As Anthony and Walshaw discuss in chapter 5, an important way in which the teacher can build students' solutions is by introducing judiciously chosen tools and representations. A second, equally important way in which the teacher can capitalise on the potential of worthwhile mathematical tasks is to engage students in justification, abstraction, and generalisation (see chapter 4), by doing which they learn to speak the language of mathematics.

The image of effective mathematical pedagogy that emerges from Anthony and Walshaw's synthesis is of teaching as a coherent system rather than a set of discrete, interchangeable strategies. This pedagogical system encompasses:

- a non-threatening classroom atmosphere;
- instructional tasks;
- tools and representations;
- classroom discourse.

To see that these four aspects of effective pedagogy constitute a system, note that the way in which instructional tasks are realised in the classroom and experienced by students depends on the classroom atmosphere, the tools and representations available for them to use, and the nature and focus of classroom discourse. And because effective pedagogy is a system, it makes little sense to think of student learning as being *caused* by isolated teacher actions or strategies. It is for this reason that Anthony and Walshaw speak of mathematical learning being *occasioned* by teaching. In using this term, Anthony and Walshaw emphasise the teacher's proactive role in supporting students' development of increasingly sophisticated forms of mathematical reasoning.

In addition to highlighting the systemic character of effective mathematical pedagogy, Anthony and Walshaw make good on the charge to develop an ecological model of learning that links what is happening in the classroom to issues of race, culture, and language, and to the school contexts in which teachers develop and revise their instructional practices. A concern for issues of equity permeate the entire report but come to the fore in the discussion of school-home partnerships that take the diverse cultures of students and their families seriously and treat them as instructional resources.

Anthony and Walshaw make it clear that it is essential to view school contexts as settings for teachers' ongoing learning. In a very real sense, these settings mediate the extent to which high quality teacher professional development will result in significant changes in teachers' classroom practices. Anthony and Walshaw's synthesis documents that mathematics instruction that places students' reasoning at the center of instructional decision making is demanding, uncertain, and not reducible to predictable routines. The available evidence indicates that a strong network of colleagues constitutes a crucial means of support for teachers as they attempt to cope with these uncertainties and the loss of established routines. Consequently, there is every reason to expect that improvement in teachers' instructional practices and student learning will be greater in schools where mathematics teachers participate in learning communities whose activities focus on central mathematical ideas and how to relate them to student reasoning. The value of teacher learning communities in turn foregrounds the critical role of the principal as an instructional leader.

Historically, teaching and school leadership have been loosely coupled, with the classroom being treated as the preserve of the teacher while school leaders managed around instruction. Recent research findings demonstrate the limitations of this type of school organisation

in supporting the improvement of teaching on any scale. These findings also indicate that principals can play a key role in supporting the emergence of a shared vision of what effective mathematical pedagogy looks like and in supporting teacher collaboration that focuses on challenges central to the development of effective pedagogy. This alternative type of school organisation is characterised by reciprocal accountability. Teachers are accountable to principals for developing increasingly effective pedagogical practices and principals are accountable to teachers to create opportunities for their ongoing learning. Changes of this type in the relations between teachers and school administrators are far reaching and might be viewed as too radical. It is, however, sobering to note that previous large-scale efforts to improve the quality of classroom instruction have rarely produced lasting changes in teachers' practices. Research into educational leadership and policy indicates that this history is due in large part to the failure to take into account the institutional settings in which teachers develop and refine their instructional practices.

The broader policy and leadership literature strongly indicates that the improvement of mathematics instruction on the scale being attempted in New Zealand is not simply a matter of providing high quality teacher professional development. It also has to be framed as a problem for schools as educational organisations that structure the institutional settings in which teachers develop and revise their instructional practices. My reading of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is that Anthony and Walshaw have distilled valuable lessons from the available research, thereby positioning New Zealand educators to succeed where others have failed.

Paul Cobb

Professor of Mathematics Education
Vanderbilt University, Tennessee

Note: The second Hans Freudenthal Medal of the International Commission on Mathematical Instruction (ICMI) was awarded to Professor Paul Cobb in 2005, "whose work is a rare combination of theoretical developments, empirical research and practical applications. His work has had a major influence on the mathematics education community and beyond."

Early Childhood Education

This Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is a 'must read' for those in the early childhood sector who want an insight into what effective mathematical pedagogy looks like in an early childhood service. The synthesis acknowledges the vital role that quality early childhood education plays in the mathematical development of infants and young children. It also provokes early childhood teachers to reflect on practice: their mathematical awareness of the environment, the depth of their mathematical knowledge, and the importance of effective teaching and learning strategies that will support children's optimal engagement in mathematical experiences. The extensive, wide-ranging research information is effectively balanced by vignettes which involve the reader in meaningful mathematical experiences that illustrate the possibilities for supporting mathematical learning. Effective distribution of the synthesis would enhance teaching and learning outcomes in early childhood services.

Faith Martin
Director, Massey Child Care Centre

NZEI Te Riu Roa

NZEI Te Riu Roa welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau, particularly as it takes for its starting point the assertion that “all children can learn mathematics”. This key message is at the heart of every teacher’s commitment to the mathematical learning of his or her students.

The synthesis recognises the complexity of teaching, particularly given the diverse learning needs of the students in our classrooms and centres and the necessity for specialised knowledge of mathematics. But the writers consistently underline the power that teachers have to make a difference: “It is what teachers do, think and believe (that) significantly influences student outcomes.”

A teacher’s role, whether in a school or a centre, includes the design of activities that help students to construct meaning and think for themselves. To achieve such outcomes, teachers need to appreciate the part that mathematics plays in the world around them, what the big mathematical ideas are, and how the concepts that they teach fit in with those ideas. They need to know how to teach knowledge and skills, how to match new learning with students’ prior knowledge, and which activities effectively encourage understanding and learning. Teachers also need to be conscious of developing attitudes and values. They need to create opportunities for their students to develop a critical eye and, in the context of this synthesis, a critical mathematical eye.

The primary purpose of the synthesis is to identify evidence that links pedagogical practice with effective mathematics outcomes for students. To achieve this, the writers have drawn on national and international research that contributes to our understanding of what works in mathematics education.

When reviewing the synthesis in its draft form, NZEI teachers were particularly pleased to read the chapter, *Mathematics Practices Outside the Classroom*, which they saw as contributing to a constructive environment and encouraging of good practice. The synthesis explores ways in which parents can contribute to their children’s mathematical development and ways in which schools can strengthen links with the home. If teachers are to successfully fulfil expectations, such links are likely to be vital. Teachers were also pleased to see the importance of school leadership recognised.

NZEI sees the *Effective Pedagogy in Mathematics/Pāngarau BES* as being of great benefit to teachers, teacher educators, and policymakers. The research identified in the synthesis, together with the case studies and vignettes, has the potential to stimulate much constructive professional discussion. To maximise its potential for teachers, it will need to be accompanied by professional learning opportunities and time for reflection and discussion in the school or centre setting.

Irene Cooper
National President
Te Manukura
NZEI Te Riu Roa

Post Primary Teachers' Association

Tēnā koutou, tēnā koutou, tēnā tatou katoa.

PPTA welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau. It is the result of a very thorough process, inclusive of the expertise of practitioners. The final report reflects and caters to their realities, and provides some very interesting and thought-provoking reading for teachers themselves, and for those involved in the pre-service and in-service education of mathematics teachers. At the same time, the research highlights the shortage of outcomes-linked research evidence specific to secondary school mathematics teaching and we hope that as a result of this BES, New Zealand researchers will step up to fill this gap.

Debbie Te Whaiti
President
New Zealand Post Primary Teachers' Association

Teacher Educators

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau succeeds in providing a systematic treatment of relevant outcomes-based evidence for what works for diverse learners in the New Zealand education system. One of the strengths of the document is the central positioning given by its authors to a broad notion of diversity.

Teacher educators, both initial and ongoing, will find that the BES is an invitation to engage—as teachers and as researchers—with a wide range of national and international studies. The document succeeds in preserving the complexity of pedagogical approaches through careful structuring and presentation. Well chosen classroom vignettes capture the essence of pedagogical issues for use in initial and ongoing teacher education. The CASEs are likely to prove particularly valuable for teachers by demonstrating how research can inform classroom practice.

The BES also presents a challenge to New Zealand researchers by identifying areas in which there is a paucity of outcomes-based evidence. Such evidence is scarce for Māori-medium mathematics classrooms. The senior secondary area is generally not well represented and a wider range of early childhood contexts needs to be investigated. The CASEs highlight for teacher educators the possibilities of writing up research projects undertaken as part of ongoing teacher education initiatives, and encourage them to gather further evidence to support practice.

The importance to mathematics education of the outcomes-based research evidence represented in this synthesis cannot be overstated. It is to be hoped that the value of the Iterative BES programme is widely recognised, and that it has the impact on policy and practice that it ought.

Joanna Higgins
Director, Mathematics Education Unit and Associate Director,
Jessie Hetherington Centre for Educational Research
Victoria University of Wellington

Māori-medium Mathematics

E nga mana, e nga reo, tēnā koutou katoa.

For the last 20 years, the teaching of pāngarau (mathematics) has played a significant role in the revitalisation of te reo Māori. The Effective Pedagogy in Mathematics/Pāngarau BES recognises the close relationship that exists between language and the learning and teaching of mathematics.

The BES identifies a range of major considerations and challenges for teachers and all those involved in Māori-medium education. The research makes it clear that mathematical outcomes for students are affected by a complex network of interrelated factors and environments, not just individual preferences or the language of instruction. By identifying the key elements in this network and discussing the relevant research, the writers have created what should prove a very useful resource.

The BES highlights the paucity of research into Māori-medium mathematics education, particularly in the area of teacher practice.

Tony Trinick
Māori-medium mathematics educator
Faculty of Education
The University of Auckland

Pasifika

E rima te'arapaki, te aro'a, te ko'uko'u te utuutu, 'iaku nei.

Under the protection of caring hands there's feeling of love and affection.

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau has drawn together a comprehensive synthesis of evidence that relates to quality mathematics pedagogical practices. Its particular strength is that it provides stimulating and thought-provoking reading for a range of stakeholders and at the same time affirms that there is no one, specific, 'quality' pedagogical approach. Rather, it directs attention to many effective approaches which make a difference for all mathematics learners. The vignettes are an added strength; they make the theoretical structures they illustrate accessible to a wider audience.

The synthesis highlights the shortage of outcomes-linked research evidence concerning quality teaching and learning for Pasifika students at all levels of schooling. It also highlights the importance of a culture of care. How this translates into quality outcomes for Pasifika students requires the attention of New Zealand researchers.

Roberta Hunter
Senior Lecturer
School of Education Studies
Massey University, Albany Campus

Chief Education Adviser, BES

The Effective Pedagogy in Mathematics/Pāngarau BES sets out to uncover and explain the links between what we do in mathematics education and what the outcomes are for learners. The result is a valuable resource that can be used to enhance a wide range of outcomes for diverse learners. These include the ability to think creatively, critically, strategically and logically; mathematical knowledge; enjoyment of intellectual challenge; self-regulatory, collaborative and problem-solving skills; and the disposition to use, enjoy and build upon that knowledge throughout life.

The BES reflects the outstanding scholarly work and professional leadership of co-authors Drs Glenda Anthony and Margaret Walshaw of Massey University. They are the first to use the new *Guidelines for Generating a Best Evidence Synthesis* and follow the collaborative development process that is central to the Iterative BES Programme. They have consulted tirelessly and responsively with a wide range of early childhood teachers, primary and secondary teachers, principals, advisers, researchers, policy workers and teacher educator colleagues from across New Zealand, and with international colleagues. The Ministry of Education acknowledges and values all these contributions—and those of the formative quality assurers, whose affirmations and challenges have been so helpful in optimising the quality and potential usefulness of this BES.

The BES celebrates and returns to early childhood educators, teachers, teacher educators and researchers a record of their professional work, highlighting the complexity of that work, and suggesting how research evidence can be a valuable resource to inform their ongoing work and that of their colleagues. From the first vignette explaining how mathematical learning can be embedded in waiata (Māori song) and dance, the vignettes bring children's learning in mathematics to life. The underlying explanations and theoretical findings have the power to inform practice in ways that are relevant and responsive to the learners in any particular centre or classroom.

The challenge now is for us all is to use this resource in ways that will support further systemic development in mathematics education, with strengthened outcomes for diverse learners. In many cases, the BES will affirm what is already happening, but it will be the points of challenge that take us forward. Individual teachers have already engaged with the BES in its draft form, and some report remarkable insights and developments in their practice. But it is only through the wider and systemic development of the conditions that support effective practice for diverse learners that improvements will proliferate and become self-sustaining. The findings emerging from the outcomes-linked professional learning and development BESs^{1,2} should be an invaluable resource in determining how to generate changed practice on such a scale.

Many teachers and early childhood educators have indicated that they want to read this BES for themselves, and to do this they need time. They need time to read, discuss and consider how they can use relevant BES findings in response to diagnostic information about the mathematical understandings of the children and young people they teach. They also need time to participate in professional learning communities. The Teacher Professional Learning and Development BES³ finds that such participation doesn't guarantee better outcomes for students, but it is a consistent feature of teacher professional learning that does have a strong positive impact.

The same BES highlights the important role that external expertise with strong pedagogical content knowledge can play in facilitating and supporting changes in practice that impacts positively on student outcomes. Such expertise can be vital in engaging teachers' theories and challenging problematic discourses. The findings do, however, caution that 'experts' need more than good intentions—in the worst-case scenario, teacher professional development can actually impact negatively on student achievement. This finding calls for careful and iterative evaluation of the effectiveness of all professional development.

The teacher education community in New Zealand has already made a foundational contribution to this BES with its engagement in the research and development reported in this BES, and its advice to the BES writers. As the Teacher Professional Learning and Development BES⁴ will show, some of our most effective professional development has been taking place as part of the Numeracy Development Projects (NDP)—with effect sizes twice those attained in England^{5,6,7}. The primary and early childhood teachers’ union, NZEI, confirms what the evaluation reports have been saying: that teachers who have been involved in the NDP value the transformational experiences this professional learning has afforded them. Two teachers from a Hawkes Bay school explained to me recently that, as a result of professional learning undertaken through the NDP, they have changed the way they work across the curriculum—they now listen more, are more diagnostic, and they place much more emphasis on children articulating and sharing their learning strategies. The dynamic, reflective, nation-wide learning community of researchers, teacher educators, teachers, and other educators created by the NDP and its Māori-medium counterpart, Te Poutama Tau, has been inspirational for BES.

If the mathematics BES is to serve New Zealand education well, the teacher education and research communities must make it a ‘living’ BES by building on the powerful insights and exemplars it makes available, addressing the gaps, and ensuring a cumulative and increasingly dynamic shared knowledge base about what works for learners in New Zealand education. To assist in this collaborative work, the New Zealand Council for Educational Research is creating a database of relevant New Zealand education theses. It has already built a database to support this document, with live links to the electronic version so that readers can quickly access either the full text or bibliographic details for some of the most helpful articles that have informed the synthesis. These links are also listed in the print version.

It is our hope that this BES will stimulate readers to let the Iterative Best Evidence Synthesis Programme know of other/new research and development that should feature in future iterations of the synthesis. Such research needs to clearly document demonstrated or triangulated links to student outcomes (see the Guidelines for Generating a Best Evidence Synthesis Iteration, found on the BES website⁸), and preferably show larger positive impacts on desired outcomes for diverse learners. We are especially seeking studies of research and development in New Zealand contexts, but we are also interested in information on overseas studies that show particularly large impacts on diverse learners. Please send details to best.evidence@minedu.govt.nz.

In the New Zealand context, where schools and centres are self managing, principals and centre leaders have a critical role to play in supporting their staff to realise the potential of this BES. The Teacher Professional Learning and Development BES indicates that, in the case of the most effective school-based interventions, principals and others in leadership roles have actively supported the development of a learning culture amongst their teachers.

For centuries, societies have required their education systems to sort children into successes and failures. Knowledge societies, such as our own, require much more. Our challenge is to ensure that all our children flourish as learners, strong in their own identities, and confident global citizens.

To achieve such goals, we need to value, build upon, and go beyond the craft practice traditions that require each teacher to ‘rediscover the wheel’. The Effective Pedagogy in Mathematics/Pāngarau BES has been designed to serve as a resource and catalyst for strengthened practice, innovation, and systemic learning. By using it, and by making learner outcomes our touchstone, we can work together to give our children a mathematics education that prepares them well for the opportunities and challenges that will be their future.

Adrienne Alton-Lee
Chief Education Adviser
Iterative Best Evidence Synthesis Programme
New Zealand Ministry of Education

-
- ¹ Timperley, H., Wilson, A., Barrar, H., & Fung, I. (to be published 2007). *Teacher Professional Learning and Development: Best Evidence Synthesis Iteration*. Wellington, NZ: Ministry of Education.
- ² Mitchell, L., & Cubey, P. (2004). *Characteristics of Professional Development Linked to Enhanced Pedagogy and Children’s Learning in Early Childhood Settings: Best Evidence Synthesis Iteration*. Wellington, NZ: Ministry of Education. educationcounts.edcentre.govt.nz/goto/BES
- ³ Ibid.
- ⁴ Timperley et al., to be published 2007.
- ⁵ Young-Loveridge, J. (2006). Patterns of Performance and progress on the Numeracy Development Project: Looking back from 2005. In *Findings from the Numeracy Development Projects 2005* (F. Ell, J. Higgins, K.C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge). Wellington, NZ: Ministry of Education.
- ⁶ Timperley et al., to be published 2007.
- ⁷ Earl, L., Watson, N., Levin, B., Leithwood, K., Fullan, M., Torrance, N., Jantzi, D., Mascal, B., & Volante, L. (2003). *Watching and learning 3: Final report of the external evaluation of England’s National Numeracy and Literacy Strategies*. London, UK: DfES
- ⁸ <http://educationcounts.edcentre.govt.nz/goto/BES>

Authors' Preface

What is a Best Evidence Synthesis in Mathematics?

A best evidence synthesis draws together available evidence about what pedagogical approaches work to improve student outcomes in Mathematics/Pāngarau. This synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme established late in 2003 by the Ministry of Education to deepen understanding of what works in education. The programme involves policy, research, and practice in collaborative knowledge building, aimed at maximising desirable outcomes for the diverse learners in the New Zealand education system.

This best evidence synthesis in Mathematics/Pāngarau plays a key role in knowledge building for New Zealand education. As a capability tool, it identifies, evaluates, analyses, and synthesises what the New Zealand evidence and international research tell us about quality mathematics teaching. It shows us how different contexts, systems, policies, resources, approaches, practices, and influences all impact on learners in different ways. Importantly, it illuminates what the evidence suggests can optimise outcomes for diverse mathematics learners.

The importance of dialogue

The development of this BES has been shaped by the *Guidelines for Generating a Best Evidence Synthesis Iteration* (Alton-Lee, 2004) and informed by dialogue amongst policy makers, educators, researchers and practitioners. Right from the very early stages of its development, the health-of-the-system perspective taken in this synthesis has ensured that we have listened to and responded to the viewpoints of a wide range of constituencies. Our interactions with these multiple communities have revealed to us the key roles that infrastructure, context, settings, and accountabilities play in a system that is functioning effectively for all its learners. Our various stakeholders have challenged us not only to produce better and more relevant educational research but to consider how this knowledge base might best be used. It is our hope that this discussion across sectors will be ongoing.

We have received a strong and positive response to the best evidence synthesis work from New Zealand's primary and post-primary teacher associations. Both have reported on how helpful the synthesis is to their core professional work. For example, the New Zealand Educational Institute (NZEI) writes: "In our view, the writers have drawn on national and international research which contributes to an understanding of what works in mathematics education; they have identified the significance of the context and ways in which to strengthen practice ... We liked the ... underpinning view that all children can learn mathematics" (p. 2). The representative for the Post Primary Teachers' Association at the Quality Assurance Day is reported as saying: "There are numerous wonderful ideas in the synthesis, and I found myself repeatedly jolted into possibilities for my own classroom resources." In addition, a group of initial and ongoing mathematics teacher educators have welcomed the "sophisticated treatment of diversity" and the way in which "the complexity of pedagogical approaches is preserved" (Victoria University of Wellington College of Education, 2006, p. 1).

Writing for multiple audiences

Our task was to make the findings of the synthesis accessible to and of benefit to a range of educational stakeholders. At one level of application, it is intended to provide a strengthened basis of knowledge about mathematics pedagogical practices in New Zealand today. The evidence it produces is expected to inform teacher educators within the discipline of mathematics education about effective pedagogical practice. At another level, the synthesis attempts to make transparent to policy makers and social planners an evidential basis for quality pedagogical approaches in mathematics. At a third level, the synthesis is expected to benefit practitioners and assist them in doing the best possible job for diverse learners in their classrooms.

Our approach to the “almost overwhelming task” (Cobb, 2006) of writing with several levels of application in mind has been to draw on both formal and informal approaches. We have offset the ‘academic’ language of the BES by including a series of vignettes that expand upon broad findings. We have received feedback from a range of sources that these vignettes bring the reality of classroom life to the fore and, in particular, do not minimise the complexities of actual practice. We hope that researchers, policy makers and practitioners alike will see in the vignettes theoretical tools that have been adapted and used by actual teachers.

The BES as a catalyst for change

This best evidence synthesis in mathematics does more than synthesise and explain evidence about what works for diverse learners. By bringing together rigorous and useful bodies of evidence about what works in mathematics, the project plays an important function as a catalyst for change. It is designed to help strengthen education policy and educational development in ways that effectively address both the needs of diverse learners and patterns of systemic underachievement in New Zealand education. It is written with the intent of stimulating activity across practitioners, policy makers, and researchers and so to strengthen system responsiveness to educational outcomes for all students.

The writers anticipate that reflection on the findings will lead to sustainable educational development that has a positive impact on learners. It will create new insights into what makes a difference for our children and young people. Reflection on the findings will also spark new questions and renewed, fruitful engagement with mathematics education. These new questions, in turn, will render the BES a snapshot in time—provisional and subject to future change.

Key features

Key features of the BES are:

- Its teacher orientation. Its view is towards a strengthened basis of knowledge about instructional practices that make a difference for diverse groups of learners.
- Its cross-sectoral approach. Its scope takes in the teaching of children in early childhood centres through to the teaching of learners in senior secondary school classrooms.
- Its inclusiveness. It documents research that reveals significant educational benefits for a wide range of diverse learners. It pays particular attention to the mathematical development of Māori and Pasifika students and documents research that captures the multiple identities held by New Zealand learners.
- Its breadth of search coverage. It reports on the characteristics of effective pedagogy, following searches through multiple national databases and inventories as well as masters’ projects and theses. It provides comprehensive information about effective teaching as evidenced from small cases, large-scale explorations, and short-term and longitudinal investigations.
- Its local character. It makes explicit links between claims and bodies of evidence that have successfully translated the intentions and spirit of the Treaty of Waitangi. It identifies research relevant to the particular conditions and contexts in New Zealand, both in mathematics education in particular and in education in general, in relation to the principles and goals of *Te Whāriki* for early childhood settings and of *The Curriculum Framework*, for teachers in English or Māori-medium settings.
- Its global linkages. It connects local sources with the international literature. It identifies important Australian and international work in the area and evaluates that wide-ranging resource in relation to similarities and differences in cultures,

populations and demographics between the country of origin and New Zealand.

- Its responsiveness to concerns about democratic participation. It heeds the concern about the development of competencies that equip students for lifelong learning. This orientation coincides with the national mathematics curriculum objective of developing those knowledges, skills, and identities that will enable students to meet and respond creatively to real-life challenges.
- Its quality assurance measures. It is guided by principles of transparency, accessibility, relevance, trustworthiness, rigour, and comprehensiveness. These principles form the backdrop to the selection and systematic integration of evidence.
- Its strategic focus on policy and social planning. It uses a health-of-the-system approach to address one of the most pressing problems in education, provide a direction for future growth, and push effective teaching beyond current understandings.
- Its provisional nature. The project is an important knowledge-building tool, creating new insights from what has gone before, and will be updated in the light of findings from new studies. The findings are, above all, 'of the moment' and open to future change.

References

- Alton-Lee, A. (2004). *Guidelines for generating a Best Evidence Synthesis Iteration 2004*. Wellington: Ministry of Education.
- Cobb, P. (2006). *Report to the New Zealand Ministry of Education on the Best Evidence Synthesis: Characteristics of pedagogical approaches that facilitate learning for diverse students in early childhood and schooling in Mathematics/Pāngarau*. Paper presented at the Quality Assurance Day, 13 February, 2006.
- New Zealand Educational Institute. (2006). *Response to the draft working document - Best evidence synthesis: Characteristics of pedagogical approaches that facilitate learning for diverse students in early childhood and schooling in Mathematics/Pāngarau*. Paper presented at the Quality Assurance Day, 13 February, 2006.
- Victoria University of Wellington College of Education. (2006). *Formative quality assurance report in terms of initial and ongoing teacher education for Best evidence synthesis: Characteristics of pedagogical approaches that facilitate learning for diverse students in early childhood and schooling in Mathematics/Pāngarau*. Paper presented at the Quality Assurance Day, 13 February, 2006.

Executive Summary

The *Effective Pedagogy in Mathematics/Pāngarau: Best Evidence Synthesis Iteration [BES]* was funded by a Ministry of Education contract awarded to Associate Professor Glenda Anthony and Dr Margaret Walshaw at Massey University. The synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme, established by the Ministry of Education in New Zealand, to deepen understanding from the research literature of what is effective in education for diverse learners. The synthesis represents a systematic and credible evidence base about quality teaching in mathematics and explains the sort of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups. It marks out the complexity of teaching and provides insight into the ways in which learners' mathematical identities and accomplishments are occasioned by effective pedagogical practices.

The search of the literature focused attention on different contexts, different communities, and multiple ways of thinking and working. Priority was given to New Zealand research into mathematics in early childhood centres and schools, both English- and Māori-medium. Personal networks enhanced the library search and facilitated access to academic journals, theses and reports, as well as other local scholarly work. The New Zealand literature was complemented by reputable work undertaken in other countries with similar population and demographic characteristics. Indices, both print and electronic, were sourced, and the search covered relevant publications within the general education literature as well as specialist educational areas. In the end, 660 pieces of research, ranging from very small, single-site studies to large scale, longitudinal, experimental studies, found their way into the report.

Key findings highlight practices that relate specifically to effective mathematics teaching and to positive learning and social outcomes in centres/kōhanga and schools/kura. The findings stress the importance of interrelationships and the development of community in the classroom. They also reveal that both the cognitive and material decisions made by teachers concerning the mathematics tasks and activities they use, significantly influence learning. The findings demonstrate the importance of children's early mathematical experiences and stress that constituting and developing children's mathematical identities is a joint enterprise of teacher, centre/school, and family/whānau.

Key findings

In this section, key findings are organised and presented according to five themes: the key principles underpinning effective mathematics teaching, the early years, the classroom community, the pedagogical task and activity, and educational leadership and centre–home and school–home links.

Key principles underpinning effective mathematics teaching

Teachers who enhance positive social and academic outcomes for their diverse students are committed to teaching that takes students' mathematical thinking seriously. Their commitment to students' thinking is underpinned by the following principles:

- an acknowledgement that all students, irrespective of age, have the capacity to become powerful mathematical learners;
- a commitment to maximise access to mathematics;
- empowerment of all to develop mathematical identities and knowledge;
- holistic development for productive citizenship through mathematics;
- relationships and the connectedness of both people and ideas;
- interpersonal respect and sensitivity;
- fairness and consistency.

The early years

Young children are powerful mathematics learners. Quality teaching guarantees the development of appropriate relationships and support as well as an awareness of children's mathematical understanding. Research has consistently demonstrated how a wide range of children's everyday activities, play and interests can be used to engage, challenge and extend children's mathematical knowledge and skills. Researchers have found that effective teachers provide opportunities for children to explore mathematics through a range of imaginative and real-world learning contexts. Contexts that are rich in perceptual and social experiences support the development of problem-solving and creative-thinking skills.

There is now strong evidence that the most effective settings for young learners provide a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities. Opportunities for learning mathematics typically arise out of children's everyday activities: counting, playing with mathematical shapes, telling time, estimating distance, sharing, cooking, and playing games. Teachers in early childhood settings need a sound understanding of mathematics to effectively capture the learning opportunities within the child's environment and make available a range of appropriate resources and purposeful and challenging activities. Using this knowledge, effective teachers provide scaffolding that extends the child's mathematical thinking while simultaneously valuing the child's contribution.

The classroom community

Research has shown that opportunities to learn depend significantly on the community that is developed within centres and classrooms. Thus, people, relationships, and classroom environments are critically important. Whilst all teachers care about student engagement, research quite clearly demonstrates that pedagogy that is focused solely on the development of a trusting climate does not get to the heart of what mathematics teaching truly entails. Teachers who truly care about their students have high yet realistic expectations about enhancing students' capacity to think, reason, communicate and reflect upon their own and others' understandings. Research has provided conclusive evidence that effective teachers work at developing inclusive partnerships, ensure that the ideas put forward by learners are received with respect and, in time, become commensurate with mathematical convention and curricular goals.

Studies have provided conclusive evidence that teaching that is effective is able to bridge students' intuitive understandings and the mathematical understandings sanctioned by the world at large. Language plays a central role. Mathematical language involves more than vocabulary and technical usage; it encompasses the ways that expert and novice mathematicians use language to explain and to justify concepts. The teacher who has the interests of learners at heart ensures that the home language of students in multilingual classroom environments connects with the underlying meaning of mathematical concepts and technical terms. Teachers who make a difference are focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics.

Mathematics teaching for diverse learners creates a space for the individual and the collective. Whilst many researchers have shown that small-group work can provide the context for social and cognitive engagement, others have cautioned that students need opportunities and time to think and work quietly away from the demands of a group. There is evidence that some students, more than others, appear to thrive in class discussion groups. Many students, including limited-English-speaking students, are reluctant to share their thinking in class discussions. Research has also shown that an over-reliance on grouping according to attainment is not necessarily productive for all students. Teachers who teach lower streamed classes tend to follow a protracted curriculum and offer less varied teaching strategies. This pedagogical

practice may have a detrimental effect on the development of a mathematical disposition and on students' sense of their own mathematical identity.

Pedagogical tasks and activities

From the research, it is evident that the opportunity to learn is influenced by what is made available to learners. For all students, the 'what' that they do is integral to their learning. The 'what' is the result of sustained integration of planned and spontaneous learning opportunities made available by the teacher. The activities that teachers plan, and the sorts of mathematical inquiries that take place around those activities, are crucially important to learning. Effective teachers plan their activities with many factors in mind, including the individual student's knowledge and experiences, and the participation norms established within the classroom. Extensive research in this area has found that effective teachers develop their planning to allow students to develop habits of mind whereby they can engage with mathematics productively and make use of appropriate tools to support their understanding.

Choice of task, tools, and activity significantly influences the development of mathematical thinking. Quality teaching at all levels ensures that mathematical tasks are not simply 'time fillers' and is focused instead on the solution of genuine mathematical problems. The most productive tasks and activities are those that allow students to access important mathematical concepts and relationships, to investigate mathematical structure, and to use techniques and notations appropriately. Research provides sound evidence that when teachers employ tasks for these purposes over sustained periods of time, they provide students with opportunities for success, they present an appropriate level of challenge, they increase students' sense of control, and they enhance students' mathematical dispositions.

Effective teaching for diverse students demands teacher knowledge. Studies exploring the impact of content and pedagogical knowledge have shown that what teachers do in classrooms is very much dependent on what they know and believe about mathematics and on what they understand about the teaching and learning of mathematics. Successful teaching of mathematics requires a teacher to have both the *intention* and the *effect* to assist pupils to make sense of mathematical topics. A teacher with the intention of developing student understanding will not necessarily produce the desired effect. Unless teachers make good sense of the mathematical ideas, they will not have the confidence to press for student understanding nor will they have the flexibility they need for spotting the entry points that will move students towards more sophisticated and mathematically grounded understandings. There is now a wealth of evidence available that shows how teachers' knowledge can be developed with the support and encouragement of a professional community of learners.

Educational leadership and links between centre and home/school

Facilitating harmonious interactions between school, family, and community contributes to the enhancing of students' aspirations, attitudes, and achievement. Research that explores practices beyond the classroom provides insight into the part that school-wide, institutional and home processes play in developing mathematical identities and capabilities. There is conclusive evidence that quality teaching is a joint enterprise involving mutual relationships and system-level processes that are shared by school personnel. Research has provided clear evidence that effective pedagogy is founded on the material, systems, human and emotional support, and resourcing provided by school leaders as well as the collaborative efforts of teachers to make a difference for all learners.

Teachers who build whānau relationships and home–community and school–centre partnerships go out of their way to facilitate harmonious interactions between the sectors. There is convincing evidence to suggest that these relationships influence students' mathematical development. The home and community environments offer a rich source of mathematical experiences on which to build centre/school learning. Teachers who collaborate with parents, families/whānau and

community members come to understand their students better. Parents benefit too: through their purposeful involvement in school/centre activities, by assisting with homework, and in providing suitable games, music and books, they develop a greater understanding of the centre's or school's programme. Their involvement also provides an opportunity to scaffold the learning that takes place within the centre or school.

Overall key findings

This Best Evidence Synthesis examines the links between pedagogical practice and student outcomes. Consistent with recent theories of teaching and learning, it finds that quality teaching is not simply a matter of 'knowing your subject' or 'being born a teacher'.

Sound subject matter knowledge and pedagogical content knowledge are prerequisites for accessing students' conceptual understandings and for deciding where those understandings might be heading. They are also critical for accessing and adapting task, activities and resources to bring the mathematics to the fore.

The importance of building home–community and school–centre partnerships has been highlighted in a number of studies of effective teaching.

Early childhood centre researchers have provided evidence that the most effective settings offer a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities.

Within centres and classrooms, effective teachers care about their students and work at developing interrelationships that create spaces for learners to develop their mathematical and cultural identities.

Extensive research on task and activity has found that effective teachers make decisions on lesson content that provide learners with opportunities to develop their mathematical identities and their mathematical understandings.

Studies have provided conclusive evidence that teaching that is effective is able to bridge learners' intuitive understandings and the mathematical understandings sanctioned by the world at large.

Gaps in the literature and directions for future research

The synthesis provides research information about effective mathematics teaching. Although the scope of researchers' studies is broad and far-reaching, a number of gaps in the literature are apparent. Research has so far provided only limited information about effective teaching in New Zealand at the secondary school level. Additionally, there is little reported research that focuses on quality teaching for Pasifika students. Few researchers in New Zealand are exploring mathematics in early childhood centres. The New Zealand literature lacks longitudinal, large-scale studies of teaching and learning. Also missing are studies undertaken in collaboration with overseas researchers. Such research is crucially important for understanding teachers' work and the impact of curricular change. The scholarly exchange of ideas made possible through joint projects with the international research community contributes in numerous ways to the capability of our local researchers.

It is important to keep in mind that, as a knowledge building tool, the synthesis provides insights based on what has gone before. A snapshot in time, it is subject to change as new kinds of evidence about quality teaching become available. Important mathematics initiatives are underway in New Zealand schools and centres. The Numeracy Development Projects, new assessment methods, projects involving information technology, and a greater focus on statistics in the curriculum are just three examples of changes that are currently taking place. All new initiatives require research that monitors and evaluates their introduction and 'take up' by centres/schools and the changes in teaching and learning that take place as a result. Such research is necessary to guide future directions in schools, educational policy, and curriculum design.

1. Introduction

Mathematics¹ is a powerful social entity. Arguably the most international of all curriculum subjects, mathematics plays a key role in shaping how individuals deal with the various spheres of private, social and civil life. The belief that mathematics is a key engine in the economy is widely shared by politicians and social planners, the corporate sector, parents, and the general public. Mathematical competencies and identities that make a valuable contribution to society are developed from specific beliefs and practices. Marshalling evidence about the pedagogical practices in centres/kōhanga and schools/kura that allow those competencies and identities to develop is a primary educational necessity. This best evidence synthesis describes those practices and provides a critical evidence base for effective pedagogical practice.

Mathematics in New Zealand

What do we know about mathematics in New Zealand schools and early childhood centres today?

Today, just as in past decades, many students do not succeed with mathematics; they are disaffected and continually confront obstacles to engaging with the subject. The challenge for those with an interest in mathematics education is to understand what teachers might do to break this pattern. Many of the problems associated with learning mathematics have little resemblance to those encountered in other curriculum areas. Typically the problems are domain-specific—solving them is not a straightforward matter of importing more general pedagogical cures.

If we cannot point to general education for students' lack of mathematical engagement, neither can we, today, point to exclusion practices whereby, traditionally, access to mathematics was considered the prerogative of a privileged few. In our inclusive society *all* students have right of access to knowledge. Precisely how teachers can enhance all students' access to powerful mathematical ideas—irrespective of socio-economic background, home language, and out-of-school affiliations—is fundamental to this best evidence synthesis.

Research has confirmed precisely what many teachers have long appreciated: that it is the classroom teacher who has a significant influence over students' learning. For example, Rowe (2004) provides evidence that when school type and the achievement and gender of students are controlled for, class/teacher effects consistently represent, on average, 59% of the residual variance in the achievements of students. Muijs and Reynolds (2001) emphasise: "All the evidence that has been generated in the school effectiveness research community shows that classrooms are far more important than schools in determining how children perform at school" (p. vii).

To a large extent, making a difference in centres and schools rests with how teachers operationalise the core dimensions of teaching. We can be sure that those core dimensions include more than the knowledge and skill that an individual teacher brings to the task. As we shall see in this synthesis, the cognitive demands of teaching, as well as the structural, organisational, management, and domain-specific choices that teachers make, are all part of the large matrix of practice. These choices include, first and foremost, the negotiation of national mathematics curriculum policy and carry over to decisions about the human, material and technological infrastructure that allow learners to achieve mathematical and social outcomes. Such infrastructural decisions involve administrators, support staff, and parents and community; they also involve the intellectual resources of curriculum materials, assessment instruments, and computational and communications technology.

Pedagogical approaches and learner outcomes

We argue throughout this synthesis for a view of pedagogy that magnifies more than what teachers know and do in centres or classrooms to support mathematical learning. And we shall

look further than improved test scores. For us, ‘best practice’ descriptions and explanations tied to high-stakes assessment don’t tell the whole story. In this synthesis, pedagogy is tied closely to interactions between people. And these interactions cannot be separated from the axes of social and material advantage or deprivation that operate to define learners. We shall see that interactions that are productive enhance not only skill and knowledge but also identity and disposition. They also add value to life and work, to the family and to the wider community of individuals (Luke, 2005).

The term ‘pedagogical approaches’ is taken as the unit of analysis and describes the elements of practice characterised not only by regularities but also the uncertainties of practice, both inside and beyond the centre or classroom. We link those practices to achievement outcomes as well as to a range of social and cultural outcomes, including outcomes relating to affect, behaviour, communication, and participation. In addition to what the teacher knows and does, pedagogy, so defined, takes into account the ways of knowing and thinking, language, and discursive registers made available within the physical, social, cultural, historical, and economic community of practice in which the teaching is embedded. Those characteristics extend beyond the centre or classroom to tap into the complex factors associated with family and whānau partnerships as well as those associated with institutional leadership and governance.

‘Quality’ or ‘effective’ pedagogical approaches are those that achieve their purposes. The exact nature of those purposes is, invariably, the subject of debate influenced by perspectives about how things should be at a given time (Krainer, 2005). Polya (1965), for example, pressed for mathematics teachers to teach people to think: “Teaching to think means that the mathematics teachers should not merely impart information, but should try also to develop the ability of the students to use the information imparted” (p. 100). Further back in time, Ballard (1915) wrote:

“We have not yet discovered the extent to which we can trust the pupils. By adopting a general policy of mistrust, by never allowing a child to mark his own, or even another child’s exercises, by making no child responsible for anybody’s conduct or progress but his own, by retaining all corrective and coercive powers in the teacher’s hands, we gain certain advantages; we simplify matters, we minimise the likelihood of abuse of authority, and we cultivate in the pupils the virtue of obedience. But we lose much more than we gain” (p. 19).

Times have changed. Today, as far as business and industry is concerned, the goals of mathematics relate to the intellectual capacities required for future employment and citizenship in a technologically-oriented, bicultural society. Prototypically for the general public, effective teaching is that which develops in students the skills, understandings, and numerical literacy they need for dealing confidently with the mathematics of everyday life. The current academic view in New Zealand is that the mathematics taught and learned in schools and early childhood centres should provide a foundation for working, thinking and acting like mathematicians and statisticians. In that view, “[e]ffective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well” (National Council of Teachers of Mathematics [NCTM], 2000, p. 16).

Making a difference for all

Irrespective of their differences, the various perspectives agree that mathematics teaching should make a positive difference to the life chances of students and should enhance their participation as citizens in an information- and data-driven age (Watson, 2006). Precisely because of the “gatekeeping role that mathematics plays in students’ access to educational and economic opportunities” (Cobb & Hodge, 2002, p. 249), it should assist students to develop:

- the ability to think creatively, critically, and logically;

- the ability to structure and organise;
- the ability to process information;
- an enjoyment of intellectual challenge;
- the skills to interpret and critically evaluate statistical information in a variety of contexts;
- the skills to solve problems that help them to investigate and understand the world.

Mathematical proficiency

These are the academic outcomes that exemplify mathematical proficiency. They include more than mastery of skills and concepts: they spell out the dispositions and habits of mind that underlie what mathematicians do in their work. The National Research Council (2001) has expanded on these strands further to suggest that proficient students are those who have:

- conceptual understanding: comprehension of mathematical concepts, operations, and relations;
- procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- strategic competence: the ability to formulate, represent, and solve mathematical problems;
- adaptive reasoning: ability for logical thought, reflection, explanation, and justification;
- productive disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

These are the characteristics of an apprentice user and maker of mathematics and are appropriated by the student through effective classroom processes.² They incorporate curriculum content, classroom organisational structures, instructional and assessment strategies, and classroom discourse regarding what mathematics is, how and why it is to be learned and who can learn it. These proficiency strands, which identify the mathematical learner within communities of classroom practice and beyond, are at the core of this best evidence synthesis.

Social, affective, and participatory outcomes

We would want to add to these academic outcomes a range of other outcomes that relate to affect, behaviour, communication, and participation. *Te Whāriki* and *The New Zealand Curriculum Framework* are useful guides to identifying these social, affective and participatory outcomes, relevant to particular age groups. The outcomes include:

- a sense of cultural identity and citizenship;
- a sense of belonging (mana whenua);
- contribution (mana tangata);
- well-being (mana atua);
- communication (mana reo);
- exploration (mana aotūroa);
- whānau spirit;
- commonly held values, such as respect for others, tolerance (rangimārie), fairness, caring (aroha), diligence, non-racist behaviour, and generosity (manaakitanga); and
- preparation for democratic and global citizenship.

Diversity

Recognising mathematics pedagogy as a key lever for increasing students' post-school and citizenship opportunities involves an important shift in thinking about students' access to learning. This changed focus is able to reveal how the development of mathematical proficiency over time is characterised by an enhanced, integrated relationship between teachers' intentions and actions on the one hand and learners' learning and development on the other. Such a focus is also able to signal how persistent inequities in students' mathematics education might be addressed, and this is crucially important in the light of recent analyses of international test data. These data reveal patterns of social inequity that cannot be read simply as a recent phenomenon but confirm a trend of systemic underachievement established over past decades (see Garden & Carpenter, 1987).

That trend points to pedagogical approaches that affect learners in unequal ways. Findings set out by the Ministry of Education (2004) reveal that New Zealand results, compared with the results of the 32 OECD countries participating in the Programme for International Student Assessment (PISA), are widely dispersed (see also Chamberlain, 2001; Davies, 2001; McGaw, 2004). While 15-year-old New Zealand students performed significantly above the OECD average and received a placing within the second highest group of countries, this positive information is offset by the fact that a high proportion of students are at the lower levels of proficiency. Data like these, signalling low proficiency levels amongst students, provide a sobering counterpoint to claims of equitable learning opportunities for diverse students.

Issues relating to student diversity are among the most complex and challenging issues facing mathematics education today. Deficit theories have tried to explain diversity by attributing the marginal performance of particular groups to the learners themselves or their impoverished circumstances. As such, these models have blamed the learner. We do not wish to ignore the fact that a considerable number of valuable interventions have resulted from this work. We point out, however, that in organising mathematical competence around the category of learner deficiency, and measuring against a 'natural', 'neutral' benchmark, the discipline offered simplistic explanations for mathematical proficiency. It could not explain why achievement comes to some learners through a hard and painful route.

Diversity is part of the New Zealand way of life. Over the next decades, our centres and schools will cater for increasingly diverse groups of learners, and these changing demographics will require a wider understanding of diversity. Diversity, however, is "a marginalised area of research, [and] is relatively undeveloped in mathematics education" (Cobb & Hodge, 2002, p. 250). In this synthesis, diversity discounts practices that stereotype on the basis of group affiliation. Instead, diversity tries to reconcile "the identities that [students] are invited to construct in the mathematics classroom" (ibid. p. 249) with their participation in the practices of home communities, local groups and wider communities within society. Characterised in this way, diversity encompasses learner affiliations with both local and broader communities. These affiliations are revealed through ethnicity, region, gender, socio-economic status, religion, and disability as well as identifiable learning difficulties and exceptional (including special) needs.

Equity

Current efforts (e.g., Cobb & Hodge, 2002) are focused on shifting from a traditional understanding of diversity towards thinking about equity. The focus on equitable pedagogical practices in this best evidence synthesis takes issue with the ill-informed belief in New Zealand society that some, but not others, are inherently equipped to learn mathematics competently. As is emphasised in the *Guidelines for Generating a Best Evidence Synthesis Iteration 2004*, equity becomes a crucially important means to redress social injustices. It is not to be confused with equality. This is because equity is about interactions between contexts and people: it is not about equal outcomes and equal approaches. Neither is it about equal access to people

and curriculum materials. Setting up equitable arrangements for learners requires different pedagogical strategies and paying attention to the different needs that result from different home environments, different mathematical identifications, and different perspectives (Clark, 2002).

Equity, as used in this synthesis, is defined not as a property of people but as a *relation* between settings and the people within those settings. Thus equity is situated rather than static and is premised on an understanding of the bicultural foundation and multicultural reality of New Zealand classrooms. Marked by fairness and justice to ‘diverse realities’ (Ministry of Education, 2004), it is responsive to the Treaty relationship held between the Crown and Māori a relationship that protects te reo (Māori language) and tikanga Māori (Māori culture) and provides assurances of same educational opportunities for Māori and non-Māori. As Cobb and Hodge (2002) argue, to understand equity, the focus needs to be not only on inequitable social structures and the ideologies that prop them up but also on how such realities play out in the everyday activity within classrooms and other cultural practices.

The synthesis is also responsive to the multiple cultural heritages increasingly brought to the centre and classroom settings and to a rapidly changing demographic profile. According to the 2001 census, by 2021, Māori enrolments at centres and schools will constitute 45% of learners and by 2050, nearly 60% of all children in New Zealand will identify as either Māori or Pasifika.³ ‘Pasifika’ is used as an umbrella term to include the cultures and ways of thinking of the Cook Islands, Fiji, Tonga, Samoa, Tuvalu, and Niue.⁴

This synthesis comes at an important point in time for mathematics education because, although the search for the characteristics of effective pedagogy in Mathematics/Pāngarau is far from new, identifying and explaining effective practice that meets the needs of *all* learners is substantially more urgent than at any previous time. Some statistics explain why this is so. Low-decile⁵ schools tend to have a greater intake of Māori and Pasifika students. In 2001, 68% of the Pasifika school population were in decile 1, 2 or 3 schools. This compares with 9.46% of the European school population. Few students from decile 1, 2, and 3 secondary schools enrol in university courses. “Middle-class students are far more likely than working-class students to experience success at school. Five times as many students with higher professional origins obtain a university entrance level bursary or better, than those from low-skilled and nonemployed families” (Nash, 1999, p. 268). These data point to the dilemmas that teachers face as they begin with the cultural and socio-economic backgrounds of their students and try to connect them to mathematics.

A few more figures drive the point home. According to Statistics New Zealand, almost one in five of all students leave school without any formal qualifications. For Māori students, the figure is one in three; for Pasifika students, it is one in four. Despite the high achievements of many Māori (see Crooks & Flockton, 2002) and Pasifika individuals, and not in any way wanting to downplay how learner performance is being raised through Māori-medium, kōhanga reo (early childhood education), kura kaupapa Māori (primary schools), and wharekura (secondary schools), the harsh reality is that average achievement, as shown in PISA and other mathematics assessments (e.g., National Education Monitoring Project), is lower for these ethnic groups.⁶

Some promising trends have been signalled in recent analyses of the 2004 Numeracy Development Project (NDP) data for 70,000 students in years 1–8 (English-medium). For all ethnic groups, achievement, as measured by the NumPA diagnostic interview, was greater than that recorded in 2003. Although the average effect size advantage for addition/subtraction was only modest (.19), the average effect sizes for the higher stages of multiplication and proportion/ratio were .4 and .43 respectively. Of particular note in the 2004 analyses was the decrease in disparities between ethnic group performances. Analysis of the 2005 achievement data in the NDP indicates that the reduction in disparities is a continuing trend (see Young-Loveridge, 2006).

To date, mathematics has been caught up in learner access to social and economic resources

and hence to future wealth and power. For mathematics education, the overriding concern is to provide equitable pedagogical access to opportunities that will develop in learners a positive mathematical disposition and enhance their life chances. Pedagogical practice that acknowledges the complexity of learners and settings allows us to move away from making gross generalisations about diverse groupings of students. Given the sociopolitical realities that shape students' constantly changing out-of-school and classroom identities, the task ahead is to change patterns of underachievement that, in the past, have been connected to a range of factors. Although there is agreement about this overarching goal, there has not been shared understanding about what pedagogy might do to achieve it.

Positioning the Best Evidence Synthesis

Against the backdrop of these statistics, the task of promoting democratic access to mathematical know-how assumes formidable proportions. How do teachers work at developing empowering approaches for learners who are polarised and disempowered by their sociocultural status? What must they do to develop an understanding of the big ideas of mathematics? How do they enhance a mathematical disposition and an appreciation of the value of mathematics in life? This synthesis, despite its best intentions, does not have ready-made practices to offer. As we shall see, the dimensions and core features of effective teaching for diverse learners are multiple. One of our most important claims is that there are no hard-and-fast rules about what methods and strategies work best. We simply do not have evidence of teaching practice that could be generalised to particular kinds of learning across all settings and across all learners. As educators, we have long known that teaching differs from one centre or classroom to another.

We caution against the tradition of identifying teacher effectiveness solely through teacher uptake of curriculum reforms or through the use of test results (Koehler & Grouws, 1992). Just as we would want to believe that reformers' visions are being realised, we have long known that teachers do not always implement them in ways that were intended by curriculum designers (Millet, Brown & Askew, 2004). We also note the limitations of craft-practice approaches to teaching that highlight teacher clarity or relative time spent on lesson components. Nor, as Alton-Lee (2005) has noted, can we say with total conviction that pedagogies customised specifically for learners with special needs produce greater achievement benefit for the learners (see Lewis & Norwich, 2000). Indeed, what we shall see is that some teaching approaches and classroom arrangements produce differential results from one setting to another. Content choices factor in too. For example, figures for NCEA results reveal that at level 2, students taking calculus recorded the lowest 'achieved' results (40%), and at level 3, 82% of statistics students recorded the highest 'achieved' results (Ministry of Education, 2006).

One thing, however, that we have gleaned from landmark studies is a set of common, underlying pedagogical principles that appear to hold good across people and settings. It is the principles upon which teachers base their practice that tend to make a difference for diverse learners. The identification of effective practice across centre and classroom settings, based on common principles, provides a rare opportunity to offer creative solutions. We do have some promising guideposts in this work. From a series of landmark best evidence syntheses (Alton-Lee, 2003; Biddulph, Biddulph, & Biddulph, 2003; Farquhar, 2003; Mitchell & Cubey, 2003), we know that practices and conditions that are respectful of the experiences of learners can make a difference to learning. We read of teachers who have provided access to learning against all odds and have done so through their belief in the rights of all learners to have access to education in a broad sense. Their work provides evidence that the effects of social disadvantage can be halted when learners encounter curricula in the classroom. Biddulph et al. (2003) have identified families/whānau and communities as key figures in bringing about academic achievement for diverse learners. Particular family attributes and processes, and community factors, as well as centre/school, family, and community partnerships, can all make a difference.

From our own discipline, we have evidence of the sorts of factors and conditions in Māori-medium schools and classrooms that raise expectations for learners' progress. Te Poutama Tau, in its responsiveness to the goals of Māori language revitalisation and empowerment, has revealed a positive effect on student achievement (Christensen, 2004). We know too that numeracy reform efforts in the UK (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997) and in this country (e.g., Thomas & Tagg, 2005; Young-Loveridge, 2005), have contributed to higher student performance.

Overview of chapters

What follows is a systematic and credible evidence base for pedagogical approaches that enhance both proficiency and equity for learners. It is drawn from research that explains the sorts of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups. It represents a first-steps approach to providing insight into new definitions of effective mathematics teaching.

The synthesis is made up of eight chapters. Chapter 2 develops a theoretical and empirical framework for the BES. We provide an overview of seminal studies that pinpoint in unique ways how quality teaching might be characterised. These landmark investigations foreground the complexity of pedagogical practice and how difficult it is to come up with universal checklists of effective teaching. From these studies, a set of guiding principles is derived alongside the theoretical framing for our work in the body of the synthesis. In offering a theoretical basis for structuring the report, we explain the notion of 'communities of practice' and the terms that we use in the synthesis. A description of how evidence-based studies of quality teaching were located, and the standards that the studies had to meet to qualify for inclusion in this synthesis, is provided in an appendix.

Chapters 3, 4, 5, 6, and 7 form the backbone to this best evidence synthesis. Covering both the early years and school sectors, these chapters take a thematic approach and include vignettes that emphasise particular characteristics of quality teaching. In chapter 3, evidence-based practices that make a difference for young learners are illuminated. This chapter on quality mathematics education in the early years stresses the formative influences at work on young children and concludes with a section analysing the transition from centre to school. Chapter 4 focuses on people, relationships, and classroom environment and explores how teachers develop productive mathematical communities of learning. Tasks and tools are brought to centre stage in chapter 5. The activities that teachers choose and the sorts of mathematical enquiries that take place around those activities are clarified. Chapter 6 explores practices beyond the classroom. It offers insight into the roles that school-wide, institutional and home processes play in developing mathematical identities and capabilities. The principles unearthed in chapter 2 and the key themes developed in chapters 3–6 converge in chapter 7 around a discussion of teaching and learning fractions. Concluding thoughts about what makes a difference in centres and schools for New Zealand learners are set out in chapter 8.

What the authors are very keen to do is clarify how patterns of inequality *can* be countered within the mathematics classroom. These explanations are not intended to be read as prescriptions of how teachers in New Zealand centres and schools should teach mathematics. Rather, by making clear the principles and characteristics underpinning effective practice, the synthesis is intended to stimulate reflection on mathematics education within and across sectors and to generate productive critique of procedures current within the discipline. Reflection and critique will make visible a new sensibility towards the multiple dimensions of pedagogical practice.

-
- ¹ Mathematics as used in this synthesis is inclusive of the statistics component in the curriculum.
 - ² By 'classroom processes' is meant the learning interactions that occur within the early learning centre and mathematics classroom.
 - ³ <http://www.stats.govt.nz>
 - ⁴ Included are the newly arrived to New Zealand, those descended from Pasifika peoples living in New Zealand for one or more generations, and those of multiple heritages.
 - ⁵ A school's decile indicates the extent to which the school draws its students from low socio-economic communities. Decile 1 schools are the 10% of schools with the highest proportion of students from low socio-economic communities, whereas decile 10 schools are the 10% of schools with the lowest proportion of these students.
 - ⁶ The actual difference may well be more marked. PISA cohorts represent students aged 15 years 3 months up to 16 years 2 months. 40% of Māori students leave school before the PISA upper age limit.
-

References

- Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington: Ministry of Education.
- Alton-Lee, A. (2005, May). *Quality teaching for diverse learners: How an evidence-based approach can help*. Keynote address to ACSA Forum, Quality teachers: Quality learning - Creating a new agenda for action by practitioners, researchers and policy makers. Melbourne.
- Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy*. London: King's College, London.
- Ballard, P. B. (1915). Norms of performance in the fundamental processes of arithmetic, with suggestions for their improvement. *The Journal of Experimental Pedagogy and Training College Record*, 3, 9-20.
- Biddulph, F., Biddulph, J., & Biddulph, C. (2003). *The complexity of community and family influences on children's achievement in New Zealand: Best evidence synthesis*. Wellington: Ministry of Education.
- Chamberlain, G. (2001). *Trends in Year 5 students' mathematics and science achievement: Results from a New Zealand study based on the Third International Mathematics and Science Study*. Wellington: Comparative Education Research team, Ministry of Education.
- Christensen, I. (2004). *An evaluation of Te Poutama Tau 2003*. Wellington: Learning Media.
- Clark, M. (2002, December). *Some problems with the emphasis on numeracy*. Paper presented at the New Zealand Mathematics Colloquium 2002. Auckland University, Auckland.
- Cobb, P., & Hodge, L. L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning*, 4(2 & 3), 249-284.
- Crooks, T., & Flockton, L. (2002). *Assessment results for Māori students 2001: Information skills, social studies, mathematics*. National Education Monitoring Report 24. Dunedin: Education Assessment Research Unit.
- Davies, D. (2001). *School entry assessment June 1997-December 2000*. Wellington: Ministry of Education.
- Farquhar, S. (2003). *Quality teaching early foundations: Best evidence synthesis*. Wellington: Ministry of Education.
- Garden, R., & Carpenter, D. (Eds.) (1987). *Mathematics achievement in New Zealand secondary schools*. Wellington: Department of Education.
- Koehler, M., & Grouws, D. (1992). Mathematics teaching practices and their effects. In D. Grouws (Ed.), *Handbook of Mathematics Teaching and Learning* (pp. 11-48). Macmillan: New York.
- Krainer, K. (2005). What is 'good' mathematics teaching, and how can research inform practice and policy? *Journal of Mathematics Teacher Education*, 8, 75-81.
- Lewis, A., & Norwich, B. (2000). *Mapping a pedagogy for learning difficulties*. Report submitted to the British Educational Research Association as part of the National Events Programme 1999. Universities of Birmingham and Exeter.
- Luke, A. (2005). Curriculum, ethics, metanarrative: Teaching and learning beyond the nation. In Y. Nozaki, R. Openshaw, & A. Luke (Eds.), *Struggles over difference: Curriculum, texts and pedagogy in the Asia-Pacific*. Albany: State University of New York Press.
- McGaw, B. (2004, June). *Australian mathematics learning in an international context*. Paper presented at MERGA 27, the Conference of the Mathematics Education Research Group of Australasia, Townsville.
- Millet, A., Brown, M., & Askew, M. (Eds.) (2004). *Primary mathematics and the developing professional*. Dordrecht: Kluwer Academic Publishers.

- Ministry of Education. (1993). *The New Zealand Curriculum Framework*. Wellington. Ministry of Education.
- Ministry of Education. (1996). *Te Whāriki. He Whāriki Mātauranga mō ngā Mokopuna o Aotearoa. Early Childhood Curriculum*. Wellington: Learning Media.
- Ministry of Education. (2004). *Guidelines for generating a Best Evidence Synthesis Iteration 2004*. Wellington: Ministry of Education.
- Ministry of Education. (2004). *Learning for tomorrow's world. Programme for International Student Assessment (PISA) 2003: New Zealand summary report*. Wellington: Ministry of Education.
- Ministry of Education. (2006). Personal communication.
- Mitchell, M., & Cubey, P. (2003). *Characteristics of professional development linked to enhanced pedagogy and children's learning and early childhood settings: Best evidence synthesis*. Wellington: Ministry of Education.
- Muijs, D., & Reynolds, D. (2001). *Effective teaching: Evidence and practice*. London: Paul Chapman Publishing.
- Nash, R. (1999). Social capital, class identity, and progress at school: Case studies. *New Zealand Journal of Education Studies*, 34(2), 267-280.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Research Council (2001). *Adding it up: Helping children learn mathematics* (Report prepared by the Mathematics Learning Study Committee). Washington, DC: National Academy Press.
- Polya, G. (1965). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Vol. II). New York: John Wiley and Sons.
- Rowe, K. (2004, August). *The importance of teaching: Ensuring better schooling by building teacher capacities that maximise the quality of teaching and learning provision Implications of findings from emerging international and Australian-evidence-based research*. Paper presented at the Australian Council for Educational Research conference, Melbourne.
- Thomas, G., & Tagg, A. (2005). Evidence for expectations: Findings from the Numeracy Project Longitudinal Study. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 35-46). Wellington: Ministry of Education.
- Watson, J. M. (2006). *Statistical literacy at the school level: Growth and goals*. Mahwah: Erlbaum Associates.
- Young-Loveridge, J. (2005). Patterns of performance and progress: Analysis of 2004 data. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 5-20). Wellington: Ministry of Education.
- Young-Loveridge, J. (2006). Patterns of performance and progress on the Numeracy Development Project: Looking back from 2005. In F. Ell, J. Higgins, K. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Projects 2005* (pp. 6-21). Wellington: Learning Media.

2. Framing the BES

Teaching mathematics is an uncertain and complex practice. Working out what teachers can do to guarantee the best possible job for diverse learners is a challenge for everyone in education. As we shall see from our aims, finding out what teachers do is our primary goal. We shall also see that our aims represent a serious attempt to address real educational problems. The synthesis is therefore both grounded in practice and strategically future-focused. Our intent is to optimise desirable learner outcomes and operationalise educational, social and pedagogical potential.

In this chapter, we develop a theoretical and empirical framework for the BES. We use a grounded approach, investigating how a number of international researchers have tried to come to grips with effective mathematics teaching. The work of the international researchers synthesised here originates from the US, the UK, and Australia as well as from our own country. The approaches the researchers take and the students they focus on are diverse and multi-levelled, spanning the early, primary, and secondary years of mathematics education.

We use their studies and findings as a backdrop to our synthesis. From their descriptions and explanations, we develop a set of guiding principles that will help us conceptualise the work ahead. We explain the framework that seems to us to provide a helpful way of exploring how teachers facilitate learning for diverse students.

Aims of the Best Evidence Synthesis

The BES has two central aims:

1. To identify and explain the characteristics of pedagogical approaches that enhance proficiency in Mathematics/Pāngarau.
2. To identify pedagogical approaches that make a significant difference for, and reduce disparity amongst, diverse learners in the early childhood and school years in Mathematics/Pāngarau.

With these two aims foremost in our thinking, in this chapter we explore the kinds of practices, contexts, policies, systems, resources, influences, and approaches that promote democratic access to powerful mathematical understandings. We do this, first in the major studies synthesised in this chapter and later in chapters 4–8, in order to draw together the sorts of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups.

A comparative perspective

The Third International Mathematics and Science Study (TIMSS) 1999 Video Study (Hiebert et al., 2003) explored classroom mathematics teaching practices. The Video Study involved seven countries—Australia, the Czech Republic, Hong Kong, Japan, the Netherlands, Switzerland, and the United States—and was undertaken as an adjunct to their participation in the comparative study of student achievement across systems of education in 50 countries. In all, 638 mathematics lessons were collected from year 8 classrooms. The videotapes captured the complexity of teaching right across the school year in all participating countries to ensure the full measure of topics and activities taking place (Hollingsworth, 2003).

Our interest in the Video Study is in exploring the characteristics of teaching that appear to enhance students' learning. One thing that becomes clear from the descriptions of mathematics classrooms provided is that there might not be a representative lesson structure within a nation. What is more unsettling for those in search of a recipe for effective practice, is that there is *no general teaching method* for generating high performance amongst year 8 students (Hiebert et al., 2003). Teaching is a process, not a technique. Whilst the pedagogical choices

made by teachers were similar (for example, nearly all teachers made use of either a textbook or a worksheet), the relative emphasis given to various instructional elements varied markedly from one high-performance country to another. What works for one teacher and one group of students will not necessarily work for another teacher with a different group of students.

In addition to the variation observed in the structure of lessons taught by teachers with different groups of students, commonalities as well as differences in mathematics teaching practice are apparent between different countries (Clarke, 2003). By way of example, take the case of two high-performance countries: Hong Kong and Japan. Like teachers in many other countries, the teachers in these two countries devoted time to reviewing old content, introducing new content and practising new content. However, in the Hong Kong lessons, these pedagogical elements were typically constructed for silent, individual learners, assigned to work on decontextualised problems. In contrast, lessons in Japan took the form of discussion and whole-class solutions to complex problems.

Given the differences in class size, lesson structure and surface-level features of the mathematical tasks, it is difficult to determine what makes a difference for students' achievement. However, the two factors that the lessons in these two high-achieving countries appear to have in common are the presentation of mathematical concepts and the level of student engagement. Teachers in these countries ensured that they did not downplay the conceptual complexity of the problems presented to students. In their presentation of content matter, they avoided simplifying problems and tended to focus on the relationships and complexities within mathematics (Watson, 2004). Teachers also worked at sustaining high levels of student engagement. When their lessons were observed closely, they were found to involve *all* learners in activities that generated knowledge. They did this by highlighting the development of reasoning and mathematical rationale and by featuring the generalisation, coherence and logical sequence of mathematical ideas. Lessons included a high percentage of mathematical problems and few repetitions of problems. Teachers in Japan, much more so than teachers from other countries, made explicit use of problems that required students to make connections between mathematical ideas, procedures and properties. This practice was in marked contrast to what teachers in Australia and the United States were observed doing in their lessons.

From the TIMSS Video Study, we can glean that student achievement cannot easily be matched with specifics such as teaching style, nor can it be linked directly with classroom organisation and student grouping. For mathematics teachers, this finding is critically important because it contradicts the common practice in some school systems of labelling learners according to particular learning styles. A case in point is the so-called kinaesthetic learner, who, consigned to a fixed-ability category, is denied engagement with the more abstract mathematical concepts offered to other groups within the class. What appears to be more critical than style or organisation for enhanced performance is the creation of a community of what we would like to call 'apprentice mathematicians', actively engaged in and responsive to the cognitive demands of the mathematics at hand. Those demands take many forms and are related in no small way to the subtopics under investigation (Rittle-Johnson & Siegler, 1998).

In an effort to pinpoint culturally specific teaching practices that engage students in the mathematics classroom, Clarke and colleagues' Learners' Perspective Study¹ (LPS) involves analysis of lesson sequences from within a range of countries. A distinguishing feature of this project is the exploration of learner practices, with the aim of assessing the effectiveness of teacher and learner actions in the promotion of particular forms of student learning. For example, Clarke's (2004) analysis of 30 lessons taught as three sequences of 10 lessons by three Australian teachers examined specific classroom interaction in relation to the achievement of teachers' pedagogical goals. Under what Clarke names as 'lesson events', it was shown that for Australian teachers, conventional practice is to 'walk between the desks'. In Clarke's view, the merits of this practice, termed 'kikan-shido' or 'instruction between desks', were multiple: first, it provided an opportunity for the teacher to interact with every individual student; second,

student participation and engagement could be monitored and assessed; third, it was clear that, as a result of their teacher's probing, prompting, and eliciting of knowledge, students became more focused and their learning was enhanced.

Effective numeracy teaching

While comparative studies paint a picture of what is happening across countries, studies involving numeracy reforms sketch out more national characterisations. In a landmark UK study of 90 teachers and over 2000 students, as well as 18 case-study teachers from the larger sample, Askew, Brown, Rhodes, Johnson, and Wiliam (1997) identified what teachers believed and understood about numeracy teaching, and related that set of data to student gains. As with the researchers of comparative studies, Askew and his colleagues found that teachers' preference for whole-class, grouping, or individualised approaches made little difference to the achievement of students. Their Effective Teachers of Numeracy project² revealed that highly effective teachers of numeracy in primary schools could be distinguished, first and foremost, by a coherent set of beliefs and understandings that underwrote their classroom work. Those beliefs related to (a) their understandings of what being numerate entailed, (b) the close interrelationship between teaching and learning, and (c) their approaches to presentation and intervention. In the moment-to-moment interchanges between teacher and students, effective teachers were shown to work at making conceptual connections between different mathematical ideas and different topics by making use of a range of symbols, words and graphics. Student discussion and teacher challenge assisted the firming up of links between ideas and the development of efficient, conceptually based strategies. Mental skills helped to sustain those links.

The tasks and activities that teachers presented were important. They engaged students as well as providing them with the challenge to think mathematically. In contrast, comparatively less effective teachers of numeracy were less likely to encourage the networking of ideas and more likely to present knowledge in fragmented and procedural form. Overriding all the characteristics of effective teaching was a firm belief in the numerical capability of the students: a belief that students could succeed.

From case studies of ten teachers in the New Zealand Numeracy Development Project (NDP), Thomas and Ward (2002) also identified a set of common characteristics of effective teachers. While pedagogical characteristics were similar to those identified by Askew et al. (1997), a notable feature was the inclusion of strong, positive relationships with students and clear expectations that students would make progress.

In a similar way to the New Zealand NDP, the Australian Early Numeracy Project sought to develop students' mathematics thinking in the early years of school through a professional development programme aimed at enhancing teachers' knowledge (Horne et al., 2002). A framework of key 'growth points' was developed to quantify student learning improvement. Using growth in student learning across all domains as an indicator of effective teaching, [Clarke and Clarke \(2004\)](#) selected six case study teachers. Using multiple data sources, the practices common to effective teachers of early years mathematics included:

- a focus on important mathematical ideas;
- structured and purposeful tasks that engage children;
- the use of a range of materials, representations, and contexts;
- making connections between mathematical ideas;
- engaging students' mathematical thinking through a variety of organisational structures;
- establishing an effective learning community;
- having high but realistic mathematical expectations of all learners;
- encouraging mathematical reflection;
- using assessment effectively for learning and teaching.

In addition to these organisational and pedagogical strategies, Clarke and Clarke (2004) identified a common set of teacher attributes that supported their quality teaching practices. They noted that the case study teachers believed that mathematics learning can and should be enjoyable, were confident in their own knowledge of mathematics and showed pride and pleasure in their students' success.

Professional development and teacher knowledge

How pedagogical practice can be changed has been the subject of several large-scale international projects, most notably in relation to numeracy reforms in the primary school sector. In England, the National Numeracy Strategy (NNS) reform programme was accompanied by a systematic and standardised national training programme that included video demonstrations of 'best practice'. The Leverhulme Numeracy Research Programme (Millett, Brown, & Askew, 2004) identified two aspects of teacher beliefs about mathematics and mathematics teaching that might be involved in deep change: beliefs about self-efficacy and beliefs about students. Teachers reported that new ways of listening and doing mathematics with their students helped them develop their own understandings of mathematics. While not identifying these changes in beliefs as indicators of deep change, the research team felt that they were of importance in themselves, providing a stronger foundation upon which to enact more effective pedagogical practices.

The objective of the New Zealand Numeracy Development Project (NDP) is to improve student learning through the development of teacher capability. A key finding to emerge from this large-scale professional development initiative is that teachers who are more successful than others at developing effective reform-based practices appear to be self-sustaining, generative learners (Thomas & Tagg, 2005). They connect what was learned in the professional development project with their own teaching and continue to reflect on and adapt what was learned as they teach (Ell & Irwin, 2006; Higgins, Bonne, & Fraser, 2004). Teachers' personal beliefs and their inclination to continually learn appear to be intrinsically related to the effectiveness of their pedagogical practices (Bicknell & Anthony, 2004).

The Cognitively Guided Instruction Project (CGI) (Carpenter, Fennema, Franke, Levi, & Empson, 1999) in the US also aims to enhance teacher capability. The intent of the CGI project is "to help teachers understand children's thinking, give the teachers an opportunity to use this knowledge in their classrooms, and give them time to reflect on what happens as a result of using this knowledge" (Chambers & Hanks, 1994, p. 286–7). The thrust of the project is that a focus on thinking and understanding, rather than coverage of content, better positions teachers to rethink and develop their own knowledge. In turn, students' learning is enhanced (Carpenter et al., 1989).

The idea that there is a close association between teachers' knowledge and student gains is compelling and has been substantiated by many other researchers. For example, in the *Study of Instructional Improvement* (Hill, Rowan, & Ball, 2005), analysis revealed that teachers' mathematical knowledge positively predicted student gains in mathematics achievement during first and third grades. An important feature of this study was the instrument used to assess mathematical *knowledge-in-use* for teaching. This task-sensitive measure confirmed the importance of pedagogically based content knowledge in specific teaching practices.

Equitable student access to learning

Understanding, explaining and addressing the processes by which inequities in mathematics continue to be regenerated is a major issue confronting educators worldwide. As we noted in chapter 1, results from international studies such as PISA and national studies and evaluations (e.g., NEMP and NDP) indicate wide disparity in mathematics achievement for students from a range of school levels. Although we have promising 2004 NDP data indicating that there has been a decrease in achievement differentials between ethnic groups, students' relative mathematical positioning based on ethnic origins remains: "Not only did European and Asian students start the project at higher framework stages, but they made greater progress than Māori and Pasifika students who started at identical framework stages" (Young-Loveridge, 2005, p. 19).

Cobb (e.g., 2002) and others in the US are currently trying to work through some of the underlying problems associated with inequitable student access to learning. Their work is motivated by a concern about the "inequitable distribution of future educational and economic opportunities" (Nasir & Cobb, 2002, p. 93). For them, equitable access is under question if certain groups of students, and not others, are required "to assimilate mainstream beliefs and values at the expense of their cultural identities" (p. 94). Foremost in their work is the idea that knowledge is necessarily social—created in the spaces and activities that individuals share in a web of economic, social and cultural differences. They explore their core themes of equity and diversity through social structures, relations of power, identity and language to reveal the ways in which a social context like the classroom is anything but neutral and to demonstrate the ways in which it reflects the wider social dynamics within society. For them, pedagogical action is more about transformative relationships than about disseminating and consuming knowledge.

In a smaller, but still significant, study in the UK, Watson, De Geest, and Prestage (2003) provide a promising way to address equitable student access to learning. The researchers found that when student proficiencies, rather than deficiencies, dominated their conceptualisations of students, teachers were able to create effective learning environments. Watson and colleagues' Improving Attainment in Mathematics Project (IAMP) specifically targeted low-attaining secondary students—students who did not know as much mathematics as their age-level peers and, perhaps more importantly, did not know how to learn mathematics. These low attainers came to be classified as such through, for example, disrupted schooling, cultural difference, social and emotional difficulties, lack of specialised teaching, limited teaching methods, low expectations, learned helplessness, reading and writing difficulties, language difficulties, physiological problems and cognitive problems.

In the research literature (e.g., Boaler, Wiliam, & Brown, 2000; Houssart, 2002; Siber, 2003; Zevenbergen, 2003), low attainers customarily learn mathematics via pedagogical strategies that are more procedural and simplified than knowledge generating. Through action research over a two-year period, the IAMP project developed pedagogical innovations with ten teachers and attempted to match teaching effectiveness with student learning, as determined by national test scores, teachers' assessments, non-routine tasks and other indicators. Driven by hopes for long-term understanding rather than short-term gains, the team sought not only to introduce students to the sorts of practices that distinguish mathematics learners from any others but also to develop student performance and interest in mathematics.

Arising naturally from the goal of developing lasting mathematical understanding was a view of teacher's work as one of ongoing support for students in developing and sustaining the sorts of practices that contribute to what mathematicians do. These practices, Watson et al. (2003) note, might be listed as: choosing appropriate techniques, generating their own enquiry, contributing examples, predicting problems, describing connections with prior knowledge, giving reasons, finding underlying similarities or differences, generalising structure from diagrams or examples, identifying what can be changed, making something more difficult,

making comparisons, posing their own questions, giving reasons, working on extended tasks over time, creating and sharing their own methods, using prior knowledge, dealing with unfamiliar problems, changing their minds, and initiating their own mathematics.

Over the two years of the project, students showed a greater willingness to engage with mathematics and work with non-routine mathematics questions. They worked longer and more successfully on activities and were prepared to confront tasks that were complex or unfamiliar to them. They became risk takers, contributing to discussions by offering alternative lines of enquiry, posing conjectures, and proving explanations and justifications. And, as a vivid indicator of the development of their mathematical thinking, they were able to generalise from given conditions.

What precisely did the teachers do that made a difference to learners who were previously mathematically disaffected? Although they shared a pedagogical intent to develop students' mathematical thinking, the teachers did not share a common pedagogical strategy to actualise that intent. Indeed, [Watson and De Geest \(2005\)](#) note that some practices “would have comfortably fitted into a typical ‘reform’ classroom; some would have comfortably fitted into a classroom in which silent textbook work was the norm” (p. 223). However, what most of the teachers agreed upon was the importance of developing and sustaining aspects of practice through which students might come to develop a positive mathematical identity. They viewed teaching as active and purposeful and focused the lesson goals on teaching and mathematical tasks rather than on impending assessment. Watson and De Geest maintain that the kinds of practices that lead to active and purposeful teaching include:

- developing routines of meaningful interaction;
- choosing how to react to correct and incorrect answers;
- giving students time to think and learn;
- working explicitly or implicitly on memory;
- using visualisation;
- relating students' writing and learning;
- helping students to be aware of progress;
- giving a range of choice;
- being explicit about connections and difference in mathematics;
- offering, retaining and dealing with mathematical complexity;
- developing extended work on mathematics;
- providing tasks that generate concentration and participation.

If teaching mathematics is about enculturating learners into mathematical thinking, then [Watson and De Geest \(2005\)](#) contend that teachers need to make explicit the conceptual networks that underpin mathematics. But more than that: they must give learners the opportunities to construct, create and navigate through a variety of conceptual structures. As teachers know, opportunities to learn depend to some extent on the systems that are set up to support learners' engagement. Norms and social relations within the systems regulate the patterns of interaction and participation, as well as specific behaviours and aptitudes such as perseverance and learning from mistakes. If these systems are inclusive and empowering, they are able to contribute to learners' development of mathematical thinking.

From these studies, it is evident that the opportunity to learn is influenced by what is made available to learners. Whatever the task or activity affords or constrains significantly influences the development of mathematical thinking. Watson (2003) contends that while there might be no observable difference in participation practices from one classroom to the next, the ways in which the learner is connected with the mathematical content can be markedly different. This important observation suggests that a focus on sociocultural practices within the centre or classroom environment is not sufficient to explain learning unless it sits alongside an explanation of how the learner is connected to the mathematics.

The same point is stressed in the Australian-based Overcoming Barriers to Learning project. This project is based on the premise that productive learning communities attend to task selection, adaptation and extension and to organisational routines, language genres, problem contexts and modes of communication. The general aim of the study is to identify elements of pedagogical practice that allow teachers to make mathematics explicit and accessible for diverse learners in the classroom (Sullivan, Zevenbergen, & Mousley, 2003). From their investigation into the pedagogical practices of teachers who taught the same content, Sullivan, Mousley, and Zevenbergen (2004) claim that there is a direct connection between the intentions that a teacher communicates, the ways in which the students respond, and the understandings that they generate about the nature of the task. They argue for an *explicit* pedagogy—one that decodes contextualised problems by responding to the language demands of the task and the explanatory demands of the mathematics under consideration, being aware of the possibility of alienating students through their differential social positionings, and anticipating diverse responses to the task under consideration.

Research studies involving centre, school, family and community partnerships provide exemplars of other ways to support the development of transformative relationships. In their Focus on Results in Math study, undertaken with schools serving mainly economically disadvantaged students across a range of states within the US, Sheldon and Epstein (2005) found that “effective implementation of practices that encouraged families to support their children’s mathematics learning at home was associated with higher percentages of students who scored at or above proficiency on standardised mathematics achievement tests” (p. 196). The project provides compelling evidence that low achievement patterns can be reversed when the learner’s access to knowledge is a prime consideration.

What these last researchers share with all the others that we have discussed in this chapter is an understanding that effective pedagogy foregrounds the learner. In these landmark studies, it is the learner’s access to mathematics and mathematical practices that becomes central. This seemingly simplistic idea goes against the grain of teaching as ‘performance’ or ‘craft’. Rather, effective pedagogy requires that teachers take account of students’ multiple identities, diverse experiences, and wide range of thinking processes. Many teachers have confronted this challenge and taught successfully—in some cases, against all odds. They have done so because they have believed that all students have the right to access mathematical culture. For these teachers, creating access to learning means creating space and time to allow mathematical thinking and acting to take place.

Conceptualising the synthesis

We have reported these cutting-edge studies, motivated by the desire to provide a starting point for conceptualising the synthesis. As we have seen in these studies, teaching is enacted in various ways by different teachers and in different classroom settings with different students. The research noted thus far takes as central the idea that teaching is a process, not a technique. In the recent International Congress on Mathematics Education (ICME-10) survey, *Relations between Mathematics Education Research and Practice*, Sfard (2005, p. 398) reported on the wish of one respondent to “systematically analyse and report ... the messy real-life classroom development” as typical of current mathematics education researchers. With regards to current research practice, Sfard also noted that the “basic type of empirical data is a carefully recorded classroom interaction, as opposed to the past attempts to document the learning of the individual student while concentrating on the result rather than on the process of teaching and learning” and that current mathematics education research “emphasizes the broadly understood social context of learning” (p. 398).

What Sfard shares with the international researchers of this chapter is an understanding that pedagogic thinking “prioritises the constitution of learning over the execution of teaching” (Hamilton & McWilliam, 2001, p. 18). Although diverse in focus, the researchers that we have reported on are committed to teaching that is less about transmission or delivery of new

knowledge and more about taking students' thinking seriously. Their commitment to students' thinking is underpinned by the following:

- an acknowledgement that all students, irrespective of age, have the capacity to become powerful mathematical learners;
- a commitment to maximise access to mathematics;
- empowerment of all to develop mathematical identities and knowledge;
- holistic development for productive citizenship through mathematics;
- relationships and the connectedness of both people and ideas;
- interpersonal respect and sensitivity;
- fairness and consistency.

We note the strong links between these and the principles that form the foundational practice in early childhood contexts, as outlined in *Te Whāriki*. Those principles are based on a recognition that classroom teaching is a complex activity. The classroom learning community is neither static nor linear. We can more usefully think of it as nested within an evolving systems network. This system might be described as an ecology in which the activities of the teacher and the students—as well as those of the centre/school and the home/community—are mutually constituted through the course of interactions. Thus, “teaching and learning coexist in a web of economic, social, and cultural differences” (Hamilton & McWilliam, 2001, p. 17).

The idea that teaching sits within a nested system draws its inspiration from Vygotskian ideas and the work of post-Vygotskian activity theorists such as Davydov and Radzikhovskii (1985). This body of work proposes a close relationship between social processes and conceptual development. This understanding forms the basis of Lave and Wenger's (1991) well-known social practice theory, in which the notions of ‘a community of practice’ and ‘the connectedness of knowing’ are central features, and in which individual and collective knowledge emerge and evolve within the dynamics of the spaces people share and within which they participate. Lave and Wenger write:

A community of practice is a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice. A community of practice is an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage. Thus, participation in the cultural practice in which any knowledge exists is an epistemological principle of learning. (p. 98)

Finding out what works for diverse mathematics learners in the system requires taking into account the processes operating at the macro-level of the system, involving policy, institutional governance, families and whānau, and communities, as well those processes found at the micro-level of the classroom. Given the scope of the system, the features of quality practice for diverse learners necessarily constitute a large matrix of practice, consisting of multiple dimensions and complex relationships between its parts. Attending to the health of the system means attending to what happens to the system's constituent parts and their interrelationships. Studies have revealed how interactions between elements within the system profoundly influence the individual's construction of mathematical ways of knowing (e.g., Boaler, 2003; Watson et al., 2003).

One of the important things that derives from these findings is the contingency of student outcomes on a network of interrelated factors and environments: outcomes are not so much *caused* by teaching practice as they are *occasioned* by those practices. Social and academic outcomes are occasioned by a complex web of relationships around which knowledge production and exchange revolve (Walshaw, 2004). Within the network, we can identify specific components within the learning environment. These we have characterised as (a) the organisation of activities and the associated norms of participation in each phase, (b) discourses, particularly norms of mathematical argumentation, (c) the instructional tasks, and d) the tools and resources that learners use. In chapter 3, which follows, the discussion is organised around these four major

components of the learning environment. The two school-based chapters in the synthesis have also been structured around these components: chapter 4 looks at the organisation of activities and the norms and discourses within the classroom; chapter 5 explores the instructional tasks and the tools and resources that students use.

¹ www.edfac.unimelb.edu.au/DSME/lps

² Numeracy in the *Effective Teachers of Numeracy* study is defined as “the ability to process, communicate and interpret numerical information in a variety of contexts” (p. 4).

References

- Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy: Final report*. London: Kings College.
- Bicknell, B., & Anthony, G. (2004). Thoughtful practice: Taking professional development a step further. *Teachers and Curriculum*, 7, 33-37.
- Boaler, J. (2003). Exploring the nature of mathematical activity: Using theory, research and 'working hypotheses' to broaden conceptions of mathematics knowing. *Educational Studies in Mathematics*, 51(1-2), 3-21.
- Boaler, J., Wiliam, D., & Brown, M. (2000). Students' experiences of ability grouping: Disaffection, polarisation and the construction of failure. *British Educational Research Journal*, 26, 631-648.
- Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.
- Carpenter, T., Fennema, E., Peterson, P., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26, 499-531.
- Chambers, D., & Hanks, J. (1994). Using knowledge of children's thinking to change teaching. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Clarke, B. A., & Clarke, D. M. (2004). Painting a picture of challenging supportive, and effective classrooms. In R. Rubenstein & G. Bright (Eds.), *Perspectives on the teaching of mathematics* (pp. 67-81). Reston: National Council of Mathematics Teachers.
- Clarke, D. (2003, April) *The structure of mathematics lessons in Australia*. Paper presented at the annual meeting of the American Educational Research Association, Chicago.
- Clarke, D. (2004). Patterns of participation in the mathematics classroom. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 231-238). Bergen: PME.
- Cobb, P., & Hodge, L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning*, 5(2&3), 249-284.
- Davydov, V. V., & Radzikhovskii, L. A. (1985). Vygotsky's theory and the activity-oriented approach in psychology. In J.V. Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp. 35-65). New York: Cambridge University Press.
- Ell, F., & Irwin, K. (2006, Feb). *Sustained numeracy project practices in two schools*. Paper presented at the National Numeracy Facilitators Conference, Auckland.
- Hamilton, D., & McWilliam, E. (2001). Ex-centric voices that frame research on teaching. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 17-43). Washington: AERA.
- Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. Washington, DC: National Center for Education Statistics.
- Higgins, J., Bonne, L., & Fraser, K. (2004). *An evaluation of the Advanced Numeracy Project 2003*. Wellington: Ministry of Education.
- Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Hollingsworth, H. (2003). The TIMSS 1999 Video Study and its relevance to Australian mathematics education research, innovation, networking, and opportunities. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, and opportunities* (Proceedings of the 26th annual conference of the

- Mathematics Education Research Group of Australasia, Vol. 1, pp. 7-15). Sydney: MERGA.
- Horne, M., Cheeseman, J., Clarke, D., Gronn, D., & McDonough, A. (2002). *Professional development and effective teaching: Developing students' mathematical thinking in the early years of school through enhancing teachers' knowledge*. Paper presented at the annual meeting of the National Council of Teachers of Mathematics, Las Vegas.
- Houssart, J. (2002). Simplification and repetition of mathematical tasks: A recipe for success or failure? *The Journal of Mathematical Behavior*, 21(2), 191-202.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Millet, A., Brown, M., & Askew, M. (Eds.). (2004). *Primary mathematics and the developing professional*. Dordrecht: Kluwer Academic Publishers.
- Ministry of Education. (1996). Te Whāriki. *He Whāriki Mātauranga mō ngā Mokopuna o Aotearoa: Early childhood curriculum*. Wellington: Ministry of Education.
- Ministry of Education. (2004). *Guidelines for generating a Best Evidence Synthesis Iteration 2004*. Wellington: Ministry of Education.
- Nasir, N., & Cobb, P. (2002). Diversity, equity, and mathematical learning. *Mathematical Thinking and Learning*, 4(2&3), 91-102.
- Rittle-Johnson, B., & Siegler, R. (1998). The relationship between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 75-110). London: Psychology Press.
- Sfard, A. (2005). What could be more practical than good research? *Educational Studies in Mathematics*, 58, 393-413.
- Sheldon, S., & Epstein, J. (2005). Involvement counts: Family and community partnerships and mathematics achievement. *The Journal of Educational Research*, 98(4), 196-206.
- Siber, E. M. (2003). *Isolating students with mathematical learning difficulties for teaching purposes: The New Zealand experience*. Unpublished master's thesis, Massey University.
- Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 257-264). Bergen: PME.
- Sullivan, P., Zevenbergen, R., & Mousley, J. (2003). The contexts of mathematics tasks and the context of the classroom: Are we including all students? *Mathematics Education Research Journal*, 15(2), 107-121.
- Thomas, G., & Tagg, A. (2005). Evidence for expectations: Findings from the Numeracy Project longitudinal study. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 21-34). Wellington: Ministry of Education.
- Thomas, G., & Ward, J. (2002). *An evaluation of the Early Numeracy Project 2001: Exploring issues in Mathematics Education*. Wellington: Ministry of Education.
- Walshaw, M. (2004). A powerful theory of active engagement. *For the Learning of Mathematics*, 24(3), 4-10.
- Watson, A. (2003). Affordances, constraints and attunements in mathematical activity. *Proceedings of the British Society for Research into Learning Mathematics*, 23(2), 103-108.
- Watson, A. (2004). Red herrings: Post-14 'best' mathematics teaching and curricula. *British Journal of Educational Studies*, 52(4), 359-376.
- Watson, A., & De Geest, E. (2005). Principled teaching for deep progress: Improving mathematical learning beyond methods and material. *Educational Studies in Mathematics*, 58, 209-234.
- Watson, A., De Geest, E., & Prestage, S. (2003). *Deep progress in mathematics: The Improving Attainment in Mathematics Project*. Oxford: Universities of Oxford and Birmingham.
- Young-Loveridge, J. (2005). The impact of mathematics education reform in New Zealand: Taking children's views into account. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 18-31). Sydney: MERGA.
- Zevenbergen, R. (2003). Grouping by ability: A self-fulfilling prophecy? *Australian Mathematics Teacher*, 59, 4, 2-7.

3. Early Years Mathematics Education

Introduction

An effective mathematics pedagogy is built on the premise that all children and students are powerful mathematics learners. The development of mathematical competencies begins at birth: “in the early months of life, [babies] are busy learning about mathematics as part of the explorations necessary to the process of becoming members of the community in which they live” (Pound, 1999, p. 3). Beginning at birth, the developments that occur in the child’s first five years represent a vitally important period of human development in their own right; they do not simply define a time to grow before ‘real learning’ begins in school (Ginsburg, Klein, & Starkey, 1998). Children develop holistically in the cognitive, social-emotional, and physical arenas, and mathematics plays a part in this development (Perry & Dockett, 2004).

This chapter is about young learners¹ and particularly about young mathematics learners in early childhood centres. Influenced by sociocultural theorists such as Vygotsky (1986) and Rogoff (1990), a model of shared learning has emerged—shared in the sense that there is valuing of children’s control but also acknowledging a significant role for the adult (Gifford, 2005). The premise that all students can be powerful mathematics learners, irrespective of age, is underscored by the New Zealand early childhood curriculum, *Te Whāriki* (Ministry of Education, 1996). Consistent with a view that early childhood provisions should offer holistic learning opportunities in the cognitive, social-emotional, and physical domains, a set of core mathematical understandings, competencies, and learning dispositions is embedded within the strands² of the document. Within the Communication – Mana Reo strand, learning outcomes for knowledge, skills, and attitudes include:

- an understanding that symbols can be ‘read’ by others and that thoughts, experiences, and ideas can be represented through words, pictures, print, numbers, sounds, shapes, models, and photographs;
- familiarity with numbers and their uses by exploring and observing the use of numbers in activities that have meaning and purpose for children;
- skill in using the counting system and mathematical symbols and concepts, such as numbers, length, weight, volume, shape, and pattern, for meaningful and increasingly complex purposes;
- the expectation that numbers can amuse, delight, illuminate, inform, and excite;
- experience with some of the technology and resources for mathematics, reading, and writing.

(Ministry of Education, 1996, p. 78)

Elsewhere in the document, there is further potential for mathematical experiences and learning, for example, “Children develop the ability to make decisions, choose their own material, and set their own problems” (Exploration – Mana Aotūroa, goal 1, p. 84).

At the outset, we want to record that there is limited empirical evidence that links quality teaching to improved educational outcomes for young children (Farquhar, 2003)—a view that is echoed by the mathematics education research community (Gifford, 2004; Perry & Dockett, 2004). To date, research appears to “have established what young children can confidently do, especially with regard to number, but we do not know much about systematically helping children to learn” (Gifford, 2004, p.100) This is an issue for educators because we now have evidence that many basic mathematical understandings are present in young children. These include enumeration, simple arithmetic, representation, problem solving, spatial skills, geometric knowledge, and some logical ability in a range of circumstances (Diezmann & Yelland, 2000; Ginsburg & Golbeck, 2004; Hughes, 1986; Kilpatrick, Swafford, & Findell, 2001; Perry & Dockett, 2002; Peters, 1992). Collectively these researchers provide evidence that young children can also demonstrate a wide range of mathematical thinking practices, including

making connections, argumentation, number sense, mental computation, algebraic reasoning, spatial and geometric reasoning, and data and probability sense.

Research studies in New Zealand and overseas suggest that while there are numerous opportunities available for children to develop mathematical ideas in early childhood education settings, these opportunities are not always utilised systematically and purposefully (Davies, 2002; Hill, 1995; Siraj-Blatchford et al., 2000; Young Loveridge, Carr, & Peters, 1995). While a radical change in mindset amongst many early childhood practitioners may be required, Gifford (2004) cautions that effective early mathematics pedagogy is also about achieving a delicate balance:

While we may want young children to start school mathematically confident, there is a danger of over-pressurising them and creating mathematics anxiety, as many adults are only too well aware. (p. 100)

With this caveat acknowledged, we provide a focused synthesis of pedagogical practices and issues related specifically to mathematics learning in early childhood education. The chapter is partitioned into sections that draw out research relevant to these key areas: task/activity pedagogical design, mathematical thinking and practices, assessment, communities of practice, teacher knowledge, and links between more formal educational settings and out-of-school settings.

The chapter concludes with a section outlining current issues around the transition from early childhood to school, in relation to mathematics pedagogy and curriculum.

Mathematical learning experiences and activities

Mathematics learning experiences should be both planned and informal/spontaneous

Young mathematics learners need opportunities and encouragement to become familiar with numbers, shapes or measuring tools before they can understand them:

They [young children] need to practise counting so that it becomes automatic before they can understand the value of numbers. When they are familiar with the shapes of building blocks, they can then use them in more varied ways and make more complex structures and patterns. Children therefore need opportunities and encouragement to become familiar and to practise if they are to investigate and generalise relationships and apply mathematics to problem solving, such as using counting to see if shares are fair. (Gifford, 2005, p. 160)

Within early childhood settings, these opportunities to learn arise from both naturally occurring, informal experiences and from planned activities. Based on findings from two large scale UK projects, Effective Provision of Pre-school Education (EPPE) and Researching Effective Pedagogy in the Early Years (REPEY), Siraj-Blatchford and Sylva (2004) concluded that the most effective settings provide both and achieve a balance between the opportunities for children to benefit from teacher-initiated group work and the provision of freely chosen, yet potentially instructive, play activities.

Everyday activities and play situations provide a source of mathematical experiences

Children's informal mathematical knowledge originates within the course of their typically occurring everyday activities. Infants, for example, learn about time and pattern through the use of rhymes and song and develop spatial skills and awareness as they move around their environment. Likewise, the everyday activities of telling time, sharing, cooking, playing games, completing puzzles, counting, estimating distances, and making music provide rich opportunities for young children to practise and develop mathematical competencies.

E rere Taku Poi

Royal Tangaere (1997, pp. 40–41) provides an example of how 18-month-old Rangī uses poi and an accompanying waiata to develop her sense of spatial concepts. The words of the song successfully direct her actions with the poi.

Original Dialogue	Translation	Nonverbal Actions
Toru whā	Three four	Hands on hip
E rere taku poi	Fly my poi	Swings poi in front of her and
E rere taku poi ki runga	Fly my poi above (me)	above her
Ki runga	above (me)	
E rere taku poi	Fly my poi	Brings poi back in front of her
E rere taku poi	Fly my poi	
Ki raro	below (me)	Swings poi down below
Ki raro	below (me)	
E rere runga	Fly above (me)	Swings up
E rere raro	Fly below (me)	Swings down
E rere roto	Fly inside	Swings into body
E rere waho	Fly outside	Swings away from body
E rere taku poi	Fly my poi	
E rere taku poi	Fly my poi	Swings poi in front of her and
Ki runga	Above (me)	then
Ki runga	Above (me)	above her

A number of thesis and research studies (e.g., Arakua, 2002; Craw, 2000, Davies, 2002; Haangana, 1999, Young-Loveridge, Carr, & Peters, 1995) contain documented examples of mathematics experiences in New Zealand early childhood centres and many others can be found within the early childhood and mathematics exemplars documents.

Play—a key component of children’s experience—provides a source of spontaneous mathematical activity, language, and thought (Davies, 1999; Ginsburg & Golbeck., 2004; Irwin & Ginsburg, 2001; Parsonage, 2001). In spontaneous play, “the practice of the community creates the potential curriculum” (Macmillan, 2004, p. 37). For example, in the following episode, the child makes a connection with a home experience and uses mathematical positioning language when describing the specific details of the snail’s movements.

A four-year-old was observing a snail crawling over pieces of celery and carrot when she said: “I’m patting him. He didn’t like it [as the snail slid off the celery]. I’m putting him on the carrot. He’s trying to get off. One of these snails was on my garbage bin at home. He’s going to fall off it in a minute. He’s going down. He can get down. (Macmillan, 2004, p. 37)

While spontaneous play in an early childhood setting enables children to engage in the practice of learning mathematics independently of the teacher, research also points to the value of shared interactions, particularly those shared with an adult. The teacher from Craw’s (2000) study reports that playing games with children is “a very good basic thing for maths ... we play lazy lion king ... so when we count, sometimes we count 1, 2, 3, 3, ... right up to 10 or in Māori up to ten ... now when I play with the picture one, I count like this with them, 10, 20, 30, 40 ... that’s how the new order comes in ... or from 0, 5, 10” (p. 10). Findings from the REPEY study (Siraj-Blatchford & Sylva, 2004) suggest that “the achievements of settings as evidenced by their cognitive outcomes appear to be directly related to the quantity and quality of the teacher/adult planned and initiated focused group work that is provided” (p. 720).

Leaving a more in-depth discussion of the role of peer and adult scaffolding until later, the following vignettes from studies of teachers within New Zealand early childhood centres, illustrate how teacher–child interactions can support the mathematical development of young children.

Aprons and Apples

The teacher in the following episode takes the opportunity to integrate spatial mathematical language while assisting a young child to put on an apron for water play:

Teacher: You look, you wear it (plastic apron) backwards if you wear it that way. So the long piece ...

Child: Okay, but I can't put this one.

Teacher: Look, the short piece and this, the long piece. The long piece goes over your front. Luke, like this and the short piece goes to your back. There! That's good—you're covered (Arakua, p. 55).

In another example, the teacher's verbal instructions for turn-taking with equipment makes explicit links with spatial concepts and the children's movement sequences: "Up on the bench, ... across the ladder, ... over the bench, ... and come underneath. Arms out for balance" (Arakua, p. 63).

In the following episode, we see the teacher and child's use of mathematical language within a discussion that is focused on the everyday resource of food:

"I've got a plate with two apples and oranges ... three apples and cut one into half and the other one into quarters, and the other one I left like that ... we were talking ... some of the children are eating and some were drawing and we had a lot of language going through ... I took two half ones and I said 'look, it's magic—two pieces'... I get the word out 'half and half make one'... then [child] said 'look: four pieces make one'..." (Craw, p. 11)

From Arakua (2002) and Craw (2000)

Increasingly children are interacting with information and communication technologies (ICT) in their everyday experiences and these experiences can also be usefully linked to mathematical activity. While research into ICT use in New Zealand early childhood centres is in its early stages (e.g., Centre of Innovation project in Roskill South Kindergarten), several studies conducted in Australian contexts are available. In a study involving 58 early childhood services, Dockett, Perry, and Nanlohy (2000) found that the use of computers added value to children's learning in terms of social and cognitive gains. Interacting within an individually appropriate learning environment over which they have some control, children gain a sense of mastery, develop representational competence, and are encouraged to create and explore in a variety of ways not otherwise possible.

Internationally, one of the most ambitious ICT projects to recognise the mathematical potential of young children is The Playground project.³ This project allows children (aged 4 to 8) to play, design, and create their own video games. Through building their own executable representations of relationships, children are able to express mathematical relationships and ideas that would normally be reserved for much older students.

In another study involving computer software, this time as a curricular programme rather than as a tool, Sarama and Clements (2004) found that young children's use of Building Blocks resulted in significant learning gains in spatial, geometric, and numeric competencies and concepts. Sarama and Clements attribute the success of Building Blocks to the use of "activities-through-trajectories"—on and off the computer—that connect children's informal knowledge to more formal mathematics. They found that teachers who understood the learning trajectories were more effective in teaching and encouraging "informal, incidental mathematics at an appropriate and deep level" (p. 188).

However, as with other studies of interventions with a mathematics focus, the potential of ICT appears to be mediated by teacher knowledge and confidence—both in ICT and mathematics (Yelland, 2005). Dockett et al.'s research (2000), referenced earlier, noted that 67% of their sample of 179 teachers in 58 early childhood centres indicated that they had never used computers with children. Given that ICT is an increasingly significant component of our lives, concerns about differential access to computers and the Internet—at home or in early childhood settings—need to be addressed (Ginsberg, Pappas, & Seo, 2001).

Activities with a mathematical focus


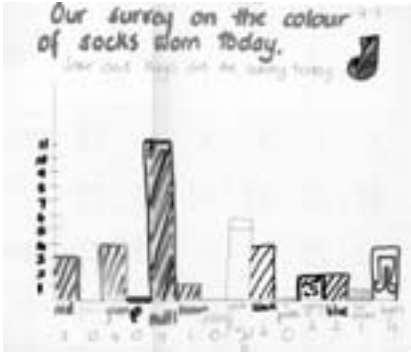
Spontaneous free play, while potentially rich in mathematics, is not sufficient to provide mathematical experiences for young children. Evidence from observational studies suggests that children's involvement in mathematical activities appears to be moderated by their own interest and prior knowledge. A US study by Tudge and Doucet (2004) of everyday mathematical activities engaged in by 39 three-year-olds showed that there was considerable variation in children's engagement. From observations of mathematics engagement at home and in centres, the researchers found no evidence to support parents' expectations that their children would more likely be engaged in mathematical activities in formal centres than at home. Davies (2002), in an intervention study involving mathematics games, also noted considerable variation in participation rates of her 10 target children. Similarly, observations of 32 case study children in the Enhancing the Mathematics of Four-Year-Olds (EMI-4s) study (Young-Loveridge et al., 1995), showed a significant variance in the mathematics focus of children.

Founded on concerns that opportunities to engage in mathematical activities may be less than optimal, early years educators recommend that centres develop an 'orientation to numeracy' by focusing on mathematical thinking within everyday activities and routines and by creating problem-solving opportunities. For instance, toddlers can be given playdough to encourage them to manipulate quantities through cutting and squashing the pieces (Perkins, 2003) and young children can explore sharing toys or food. In the following vignette, we see how Jake initiates a survey of bags in lockers. It appears that the activity is directly related to an earlier learning experience in which the teacher and Jake and Hugo produced a bar graph of the colour of socks worn on one day. In the Learning Story, we see how Jake attempts to collect the data in a systematic way, previously modelled by the teacher.

Jake's Learning Story

Jake works to a plan. He is systematic and likes to complete a job. He has become very involved in surveys and likes to discriminate, sort, match, count, and record.

<p>Belonging Mana whenua</p>	<p>Taking an interest</p>	<p>Jake arrived, walking up the ramp, saying he would like to do a survey on bags. He came to me and we talked about how he would need to go about this. Jake thought this was a good idea for a survey as he didn't have to ask anyone any questions!!</p>
----------------------------------	---------------------------	---

<p>Well-being Mana atua</p>	<p>Being involved</p>	<p>Jake had a clip board and worked on the yellow table. He drew bags and coloured them. "Look this one doesn't have a handle"</p> 
<p>Exploration Mana aotūroa</p>	<p>Persisting with difficulty</p>	<p>I asked if he was going to have multi coloured. Jake explained that there were no multi coloured crayons. I suggested he go and have a look at the sock graph to see how I had made multi colour. Came back still stumped. Finally I asked if he needed help. I showed him how I drew lines of different colours. At the table the other children discussed what made multi coloured. Two colours were two tone, so you needed three or more to be multi coloured. Jake also drew a big bag with a cross through it for 'no bags'.</p> 
<p>Communication Mana reo</p>	<p>Expressing an idea or feeling</p>	<p>He worked through looking at the lockers. Came to get me. He was not sure if he had got them all and said there were a lot with no bags. I asked if he had started from the top and worked along.</p> <p>Jake looked horrified. "I started from the bottom and worked along".</p>
<p>Contribution Mana tangata</p>	<p>Taking responsibility</p>	<p>I explained that was fine, I only meant had he worked in a line to make it easier and it didn't matter where you started. Jake was fascinated that there were bags the same. Jake is absorbed in thinking up ideas of what he would like to survey. He seems totally in charge of this!</p>

From Ministry of Education (2007)

The provision of more challenging and complex mathematical experiences has been investigated in a range of intervention studies. Introduced activities with a mathematical focus include games (Davies, 2002), weekly themes/projects (Sophian, 2004), problem-solving adventure stories (Casey, Kersh, & Young, 2004), imaginative kits (Macmillan, 2002), books (Young-Loveridge et al., 1995), and technology (Sarama & Clements, 2004). Results of these studies highlight the benefits of increased opportunities for students to access mathematical ideas, and increased teacher awareness of and confidence with mathematics.

In New Zealand, the landmark Enhancing the Mathematics of Four-Year-Olds (EMI-4s) intervention project (Young-Loveridge et al., 1995), focused on ways to enhance the mathematical understanding of four-year-olds. This large-scale project involved four kindergartens, one of which was a control. During a three-month period, the researchers worked collaboratively with teachers to increase children's access to mathematical experiences. The intervention within the centres was two-fold. First, teachers' awareness of mathematics in everyday contexts and of children's interests⁴ was heightened through collaboration with the researchers. Second, additional resources with a mathematical focus (e.g., books, games, dice, calculators, calendars, and measuring tools) were integrated into the centres' programmes. In addition, efforts to increase the amount of display material involving numeracy resulted in increased talk about numbers and quantities. Improved performance, when compared with that of children attending a kindergarten without the intervention, was credited to access to additional and appropriate activities and equipment, and to a reported increase in teacher awareness of mathematics:

Much more aware. When I [teacher] am reading a story I stop when there is counting to be done, and do it slowly with them [children] and we all join in. ... I am aware of it in other areas—outside, waiting for turns, or the number of children doing something ... I am probably not aware of brand new things happening, but I think I am aware of things that have always been happening and focusing in on these. (p. 124)

In contrast to the more holistic approach of the EMI-4s study, interventions in the US typically seek to identify how *specific* mathematical competencies can best be taught. Griffin (2004) reports on the pre-K–2 mathematics programme, Number Worlds. Developed to teach conceptual structure for number, the programme claims to build upon children's current knowledge through the provision of multilevel activities, to utilise activities that follow a natural developmental progression, and to teach computational fluency as well as understanding. Activities include contextual problems using multiple representations of number: a group of objects, a dot-set pattern, a position on a line, a position on a scale, and a point on a dial. Evaluation reports found that the children from the intervention study made significant gains in conceptual knowledge of number and in number sense when compared with matched-control groups who received readiness training of a different sort.

Starkey, Klein, and Wakeley's (2004) intervention programme, like Number Worlds, targeted children from economically disadvantaged families in a US context. The intervention strategy was similar to that employed in the EMI-4s project: the introduction of targeted activities with a mathematics focus, including computer-based mathematics activities, and teacher professional development. In addition, parents and children in the intervention group attended a series of three home mathematics classes where activities were presented and strategies for dyadic engagement discussed. The significant socio-economic status (SES)-related gap in mathematical knowledge found at the beginning of the pre-kindergarten year was decreased following the intervention year.

Sophian (2004) reports on another US intervention designed to address the perceived disadvantage—in terms of levels of mathematical knowledge—of children from low-income families. Implementation of an experimental mathematics curriculum developed for Head Start included teacher supports and home activities that corresponded to activities presented in the centres. However, unlike the previous interventions that focused on number, this curriculum intervention focused on the concept of unit as it applies to enumeration, measurement, and the identification of relations among geometric shapes. Measurement activities involved measuring

the same quantity using different units, an approach not typically found in other preschool curricula. For example, children made a row of prints of their own hand on strips of cloth and then used their strip to measure various objects. They then explored what happened when the teacher measured with prints of her own (larger) hand. Part-part-whole relationships, typically associated with number bonds, were developed through an examination of partitioning area. Children who received the mathematics intervention outperformed both no-intervention control children and children who had participated in a literacy rather than a mathematics intervention. As with the other intervention studies, increased teacher understandings about mathematics and increased teacher expectations about what children were able to achieve and understand in relation to mathematics were noted.

What is interesting about this study and related research projects in primary schools (e.g., Dougherty's Measure Up Curriculum) is that it challenges the dominance of counting and number as the entry point for mathematics development. The programme differs from most other early childhood programmes in that measurement is a perspective that pervades the entire curriculum. The Head Start curriculum views mathematics learning as "primarily a matter of learning to reason effectively about quantity (particularly, for young children, tangible, manipulable, physical quantities) and only within that broader objective as a matter of learning about numbers" (Sophian, 2004, p. 76).

'Patterning' is another related mathematics skill that is spontaneously evoked in young children.

A two-year-old chants to herself in the bathtub, "splish, splish, oh-oh, splish, splish, oh-oh" while simultaneously squeezing the water out of a rubber toy. With each "splish" she squeezes the toy and then declares "oh-oh" when no water comes out on the third and fourth squeeze. She then submerges the toy to refill it and repeats the event. (Schwartz, 2005, p. 1)

Focusing on children's interests and play situations, Australian researchers Papic and Mulligan (2005) developed an intervention programme built on children's existing ideas about pattern. Evaluation of the intervention programme involving two early childhood centres (target and control) reported a sustained positive impact on the mathematical development of the intervention children (who represented a range of ability), both during the six-month period of the study and 12 months later, at the end of the first year of formal schooling. Based on strong correlations with children's ability on patterning tasks and other numeracy assessments at the end of the first year of formal school, the researchers argue for the existence of "strong links between a child's ability to pattern and their development of pre-algebraic and reasoning skills" (p. 615). (See also Mulligan and colleagues' research in chapter 5.)

Appropriate challenge

A recent international discussion group on the mathematical thinking of young children (Hunting & Pearn, 2003) noted that young children are more capable than current practices suggest. The expert group called for the provision of more challenging early education programmes. This recommendation is in accord with the common finding noted in the intervention research programmes reviewed in this chapter; namely, that participating teachers were often unaware of the need to cater for children's interests and mathematical abilities and to engage children in challenging learning experiences. Participation in the intervention programmes challenged teachers' expectations about what mathematics young children could learn and understand.

As noted in the REPEY study⁵ of centres in the UK, many opportunities for sustained, shared cognitive engagement were missed and the cognitive challenge involved in teacher-child interactions was sometimes less than optimal. Analysis of high-cognitive challenge activities in the REPEY uncovered an interesting pattern:

In excellent settings the importance of staff members extending child-initiated episodes is very clear; just under half of child-initiated episodes observed as

high challenge involved interventions from a staff member which extended the child's activities. The preponderance of staff extension in child-initiated activities appeared to be unique to the three highest performing ('excellent') settings. ... Analysis showed that the most common critical point ('lifting the level of thinking') occurred when a practitioner 'extended' a child-initiated episode by scaffolding, thematic conversation or instruction. (Siraj-Blatchford & Sylva, 2004, p. 723)

The following vignette provides an example of sustained, shared thinking (though science-based) from the REPEY data.

Bubbles

- Boy 8: [Who has been watching various items floating on water.] Look at the fir cone. There's bubbles of air coming out.
- Nursery Officer: It's spinning round. [Modelling curiosity and desire to investigate further.]
- Boy 8: That's 'cos it's got air in it.
- Nursery Officer: [Picks up the fir cone and shows the children how the scales go round the fir cone in a spiral, turning the fir cone round with a winding action.] When the air comes out in bubbles it makes the fir cone spin around.
- Girl 2E: [Uses a plastic tube to blow into the water.] Look bubbles.
- Nursery Officer: What are you putting into the water to make bubbles? ... What's coming out of the tube?
- Girl 2E: Air.

The episode illustrates the power of the skilled partner, in this case the Nursery Officer, to understand and build on what it is that Boy 8 understands or does not understand.

From Siraj-Blatchford and Sylva (2004)

Based on the findings of the EMI-4s study, Young-Loveridge and colleagues also concluded that effective teachers were those who were able to pick up on the children's mathematical ideas and extend them, and, in accord with the REPEY findings, the researchers also noted frequent missed opportunities for both interaction and extension. Many of the mathematics activities were suited to children with average numeracy levels; activities more appropriate for novices and experts were noticeably missing. Given the variability of children's entry knowledge, finding the appropriate level of challenge was seen to be a critical factor in catering for the needs of diverse students.

The intervention phase of the EMI-4s study detailed how activities could be successfully adapted to provide a suitable level of challenge. Social and physical adaptations to games included avoiding standard turn-taking by providing each child with a dice so that they could play simultaneously and the use of large-scale resources such as outdoor number tracks, instead of baseboards and counters, to reduce the need for fine motor skills. Cognitive adjustments to games to suit the skills of children who could count only to two or three included the use of dice showing patterns of only one, two and three. A suitable challenge could be provided for a more expert player by suggesting that they add or subtract the scores on two dice to determine a player's move. It was noted by the teachers involved in the intervention phase that their increased awareness of mathematics was the key to the provision of appropriate activities and to their ability to capitalise on spontaneous mathematical interactions with children.

While provision of sufficient challenge is seen as a priority, Carr et al. (1994) note that too much challenge—determined by a combination of familiarity of context, meaningfulness of purpose, and complexity—also brings its own set of problems. If too many activities are beyond the difficulty limits of children and they are recognised as mathematical, young children may begin to avoid or ignore all mathematical tasks.

Assessment

Assessment in the early years is seen as an integral part of learning: “Assessment sits inside the curriculum, and assessments do not merely describe learning, they also construct and foster it” (Ministry of Education, 2004e, p. 3). The New Zealand early childhood exemplar document *Kei Tua o te Pae* describes assessment for learning as ‘noticing, recognising, and responding’: “Teachers *notice* a great deal as they work with children, and they *recognise* some of what they notice as ‘learning’. They will *respond* to a selection of what they recognise” (Ministry of Education, 2004e, p. 6).

Research on assessment practices in general indicates that effective pedagogy is informed by contextual knowledge of children’s learning and mathematical understanding (Alton-Lee, 2003). We know that young children are capable of building rich sources of informal and formal mathematical knowledge. We also know, however, that competence levels vary (Young-Loveridge et al., 1995) and consequently young children need learning experiences that match their current understandings (Farquhar, 2003).

In some cases, variability of children’s mathematics knowledge has been attributed to a lack of opportunity to learn rather than an inability to learn. In the EMI-4s study, Young-Loveridge and colleagues (1995) observed that some capable children who were very quiet managed to avoid bringing their expertise to the attention of their teachers. Moreover, children with low levels of expertise but who were outgoing and confident were sometimes assumed to have much greater proficiency than they actually did have. For example, a child (coded T) was confident in a range of areas but not mathematics. Prior to the intervention period of the study, his teachers assumed that his ‘mistakes’ arose from ‘just kidding around’. During the intervention phase, the combination of the teachers’ increased awareness of mathematics and T’s increased opportunities to participate in mathematical activities revealed that although he was obviously interested in number, his skills were in fact limited. Given this information, his teachers were able to support his mathematical development more effectively.

Awareness of children’s mathematical understanding

Early and appropriate assessment enables teachers to gain information for teaching and early intervention where necessary. Sarama and Clements (2004) contend that group interactions within early childhood settings often hide individual needs. They argue for more research that identifies effective ways of combining group work with individual assessment and teaching. Systematic observations of children, shared experiences, involving children as informants, and effective communication between teachers and families are all assessment practices that early childhood teachers have found to be effective (McNaughton, 2002).

In contrast to the more formal diagnostic testing that emphasises developmental progressions, assessment in early years mathematics needs to not only look at the individual but also acknowledge the role of the context and the child’s interactions with others (Fleer, 2002). In her case study of how one teacher in a New Zealand early childhood centre understood her children’s mathematical thinking, Craw (2000) documents the teacher’s effective practice as follows:

In order to do this she talks about “following children around: and endeavouring to engage in “play with the children”, “talk to them” and try to interpret ... “what maths” before something could be “set for the children” that would extend their learning in ways that will motivate children’s thinking to enable “eye openers in the sense that the brain is activated” and the children are able to “see things and to observe”. (p. 9)

The potential of children’s portfolios to highlight literacy and numeracy learning is advocated by Hedges (2002). She claims that a wider perspective on pedagogical documentation, including evidence of content learning, will both contribute to knowledge of children’s learning and provide a platform for children and parent communication. Similarly, Learning Stories, a formative assessment practice developed by Carr (1998), recognises the alternative strengths

of the child, and as such, offers promise for supporting personalised teaching approaches for diverse learners of mathematics (Education Review Office, 2004).

Tom's Learning Story

The following example of an abridged text captures an episode of learning in a New Zealand centre. The story highlights learning dispositional influences, key competencies and specific mathematical skills of measuring, understanding symbols, and communicating mathematical ideas, displayed within the context of a measuring activity.

Belonging Mana whenua	Taking an interest	Tom held up a long piece of dough he had squeezed from the piping equipment and explained. "Look Rosie – it's sooo long!"
Well-being Mana atua	Being involved	"Yes, you're right Tom – it sure is! Let's get a ruler and measure it to see how long it really is," I suggested. Tom placed his dough strip along the tape measure.
Exploration Mana aotūroa	Persisting with difficulty	"Can you see the numbers Tom – they tell you how long it is," I explain. After studying the numbers carefully Tom announced "19 long!" "Yes, 19 centimetres," I add.
Communication Mana reo	Expressing an idea or feeling	"I'll make another one – but even longer this time. Look this one is ... 22 cm" he continues.
Contribution Mana tangata	Taking responsibility	"Wow, can you make a strip as long as the ruler – 30 cm long Tom?" After much squeezing and slight adaptation, Tom successfully makes the strip reach from one end to the other. "Look – it's 30 cm long now!"

In addition to highlighting the mathematical nature of Tom's activity (measuring, understanding symbols, and mathematical exploration) the Learning Story portrays Tom's interest in measuring, sustained involvement in the activities, persistence with difficulty when making the strip as long as the ruler, and verbal expression of his ideas. In terms of key competencies, Tom is involved in logical thinking and adapting ideas, and making meaning in relation to tools, symbols and language for measuring. The Learning Story is also able to capture Tom's mathematical discussion with his teacher, and engagement with the wider culture community of mathematics users.

From Peters (2004)

A succession of learning stories can effectively demonstrate a child's progress in a range of contexts, the nature of the strategies and dispositions involved, and the degree of increasing mathematical development. For example, the learning story, *Jaydon's Towers* (see *Assessment for Infants and Toddlers: He Aromatawai Kōhungahunga, Tamariki*, Ministry of Education, 2004b, pp. 12–14), documents Jaydon's spatial development within a seven-month period, based on a range of sorting and classifying and construction activities.

Shared purpose and interests

Building on a child's understandings and interests requires teachers to not only assess a child's developing mathematical knowledge but also to understand and respond to the cultural and social perspective of the learner (Hedges, 2002; Macmillan, 2004; Watego, 2005). Supporting adults need to build mathematical opportunities into those contexts that are familiar and appealing to young children—a factor that the EMI-4s study found particularly significant for lower attaining children.

In attending to children's social viewpoint, researchers have found that young children often have their own social purpose for mathematics, especially number. Mathematical ideas that are “genuinely powerful for young children have much more to do with the processes used to interact with and do mathematics than with particular items of mathematical knowledge” (Perry & Dockett, 2002, p. 88). As such, young children rely on intuitive concepts and techniques for solving mathematical problems. For example, they may be aware of numbers and quantities but not yet use the number system and other formal mathematical tools. Having a collection of strategies to resolve situations that are relevant to them is much more important than knowing ‘correct’ mathematical terminology or being able to recite basic addition facts.

As noted in chapter 5, the resulting learning experience may sometimes be mediated by a purpose different from that intended by the teacher. Munn's (1994, cited in Worthington & Carruthers, 2003) study of pre-school children's counting ability noted a clear distinction between the purposes that children ascribed to counting and the purposes ascribed by adults. For example, a learning experience such as a block-based activity, designed with the aim of providing a rich problem experience, may have its purpose subverted to one of entitlement as children hoard, guard (and count) their supply of blocks. In the EMI-4s study, children's purposes included ritualised number use (e.g., rhythm and repetition), establishing status, establishing entitlement, timing, patterns, orderliness, labelling, playing with numbers (just for fun), and exploring quantity to solve a particular problem (e.g., measuring, designing, recording, locating, and playing games).

Differing Mathematical Purposes

Elizabeth sees number as poetry. Her main reason for using number is for the purpose of rote-counting and for status (ages and birthdays are of great interest to her). She is an enthusiastic rote-counter, and does not appear to be interested in using number for counting things or to find out ‘how many’. Her interest in stable rote sequences extends to knowing the alphabet and to using the rhyme ‘eenie, meenie, minie, mo’ to count her sandwiches at lunch time. She enjoys repeated refrains and choruses in stories and songs.

Todd is more of a pattern maker. He enjoys drawing multiple copies of a particular object (triangles squares, aeroplanes, cars), cutting them out and rearranging them. He doesn't use number very often, but he uses it for a wider range of purposes than does Elizabeth: for making patterns, building with blocks, keeping score at badminton. His rote-counting skills are uncertain, but when the numbers are small he can use number in problem-solving activities.

From Carr, Peters, and Young-Loveridge (1991)

Children's different cultural backgrounds engage them with different experiences and expectations of different packages of mathematical purposes. For example Ha'angana (1999) provides a description of how the activity of making of a *kahoa* supported a young boy's patterning and counting skills: "He strung together 22 pieces. He made a pattern with five colours. Each time he completed the five counts, he counted the next five pieces before he continued to string them together" (p. 34).

The usefulness of mathematics in providing an authoritative voice within contexts of sharing and entitlement is another purpose that arises in young children's lives.

Who Should Be Mum?

Two young girls aged four-and-a-half years are playing in the family area of their centre. Their efforts to resolve a conflict situation are based on a mathematical logic, which while not necessarily coherent to adults, appears coherent to the children. Both girls have determined that size, as determined by height, is the critical factor in deciding who should be 'Mum'.

Stella placed her hands on her hips and sighed. Jane adopted a similar stance and called loudly, "I'm the mother, I'm the mother." She then moved closer to Stella, stood straight, and added "I'm the mother! See, I'm bigger than you!" "No, I'm higher," replied Jane, "I'll show you." She stood right next to Stella and said, "Look! See, I'm bigger!" Stella looked, and stretched as high as she could. "And I'm big!" Jane looked again and complained, "Don't stand on tippy toes, that's not fair!" When Stella did not react, Jane added, "I'm gonna see my Daddy."

From Perry and Dockett (2002), p. 92

Sfard's (2005) extended account of two four-year-old children responding to a parent's requests for quantitative comparisons also illustrates that "if we are unable to see the children's reasons, it is likely to be the result of our tendency to interpret their utterances the way we would interpret our own" (p. 242). Her research reiterates that children's use of words may be dramatically different from that of adults. Awareness that children will interpret activities according to their own social and cultural experiences enables teachers to more effectively select and respond to opportunities to learn and, where appropriate, to make the purpose of mathematical learning explicit.

Communities of practice

Effective pedagogy assists children to access powerful mathematical ideas through the development of appropriate relationships and support. As we have seen earlier, research findings indicate that positive assessment and learning outcomes are closely associated with adult-child interactions that involve some element of sustained, shared thinking. These collaborative and responsive interactions between children and adults or more expert peers involve scaffolding and appropriation as well as providing opportunities for young children to develop metacognitive strategies and knowledge.

Scaffolding and interactions

A shift towards a consideration of Vygotskian principles relating to the social mediation of knowledge has prompted early childhood educators to focus not only on what it is that children are capable of on their own, but also on what they are capable of achieving with the assistance of more knowledgeable others, through scaffolding (Jordon, 2003). When working within the child's 'Zone of Proximal Development'⁶ sociocultural perspectives:

emphasise that children's higher mental processes are formed through the scaffolding of children's developing understanding through social interactions with skilled partners. If children are to acquire knowledge about their world it is crucial that they engage in shared experiences with relevant scripts, events, and objects with adults (and peers). (Smith, 1999, p. 86)

Learning with the support of peers is one consequence of social interactions in early childhood settings and within communities. Young-Loveridge et al. (1998) provide the following account of a capable four-year-old girl, A, helping child N play the computer game.

A used a variety of methods to help N with the game. She took N's hand and placed it on the appropriate numeral key, demonstrated how to count the objects on the screen and also translated the set on the screen into an equivalent set of fingers, perhaps intuitively sensing that it would be easier to count a set of large concrete object such as fingers, as opposed to small symbols on a screen. N's one-to-one interactions with A, as she struggled to solve the mathematical problems posed by the computer, were the most cognitively challenging episodes in the entire observation period. (pp. 88–89)

When considering interactions with children and teachers, Siraj-Blatchford and Sylva (2004) found that effective 'pedagogical interactions', as distinguished from the 'pedagogical frame' (the behind-the-scenes aspects of pedagogy, which include planning, resources, and establishment of routines), were significant indicators of children's performance. Effective pedagogical interactions contain elements of 'sustained, shared thinking' and contrast with less positive interactions involving monitoring of behaviour or engagement. Siraj-Blatchford and Sylva (2004) noted that the level of thinking was most likely to be enhanced when a practitioner "extended a child-initiated episode by scaffolding, thematic conversation or instruction" (p. 723).

Mathematics in the Sandpit

Outside in the sandpit, the teacher, working with two boys, simultaneously scaffolds and extends the children's thinking.

- Teacher: How many children have been making this? [referring to hole in the sand].
- Child: Three.
- Teacher: How many have been making it? One, two and yourself, three. That's right three children have been making this and you have been digging this hole for a long time because it is a very big hole.
- Child: Yeah! One hundred. One hundred minutes.
- Teacher: One hundred what?
- Child: Yeah, lots of minutes that is why we need ... take a long time to do this.
- Teacher: One hundred minutes? That is a lot of minutes isn't it?

From Arakua (2002)

For effective sustained, shared teaching episodes within a play situation, scaffolding needs to extend the child's thinking while simultaneously valuing the child's contribution, allowing the child to retain control of the play. Through their actions and words, adults can encourage children to persevere with a problem, think about it in different ways, and share possible solutions with peers and other adults. They can challenge children to extend their thinking or the scope of their investigations. Based on an analysis of transcripts of children and adults playing with numeracy activities⁷ in early childhood centres, Macmillan (2002, p. 4) provides the following examples of teaching strategies that are responsive to a child's developing sense of identity as learner.

Clarifying/elaborating:

- 'This one here's number one. Can you find your number one?'
- 'That goes with the tree A's holding.'

Recognising/appreciating:

- 'Wow! Look at all that matching!'
- 'You're doing a great job there. You have all the trees there.'

‘They are beautiful, aren’t they?’

Confirming:

‘There’s only one, is there?’

‘You can do it wherever you want to, A.’

‘It looks like a real one.’

Encouraging reflection by asking assisting/checking questions [link with metacognition]:

‘Do you need more?’

‘Does that match?’

‘Do we need to put them all in one dish?’

Pretending not to know the answer:

‘Truly, I can’t count!’

Creating relevance by making links with the child’s current knowledge:

‘That’s nearly as old as you.’

‘There’s a fat brick and a skinny brick.’

Modelling curiosity:

‘I wonder if mine’s a circle?’

Inviting imaginative involvement:

‘Looks like a wiggly worm.’

Inviting participation by offering choice:

‘How many do you think I’ll need?’

‘Who’s going to put a card out first?’

‘Where do you want me to put this one? In here, near the ladybug?’

Inviting participation by offering challenge:

‘Let’s count the red ones you’ve used.’

‘How many do you think I’ll need?’

Exemplifying a proactive pedagogical role, these responsive interactions contribute to an environment that stimulates exploration and offers opportunities for children to question and challenge their understanding.

Metacognition

Social interactions during early learning also communicate to the child messages about his or her intellectual capabilities. Learning about how to meet success and failure and how to plan for the future contribute to the early development of a child’s metacognitive knowledge base.

The *Quality Teaching for Diverse Students in Schooling: Best Evidence Synthesis* (Alton-Lee, 2003) identifies the development of metacognitive knowledge and strategies to support appropriate learning orientations and self-regulation as a significant factor for student achievement. While young children clearly have less metacognitive ability than older children, the development of self-regulation skills and related metacognitive awareness is seen as a significant goal within early childhood education. Cullen’s (1988) research demonstrates that the provision of opportunities for young children to interact with peers and to practise metacognitive-like activities such as planning, monitoring, reflecting, and directing enhances children’s development of self-regulatory skills.

Turn Taking

The following excerpt from a kōhanga reo illustrates children’s self-monitoring of their progress in a game-playing episode. Realising that they have got lost, they decide to return to the beginning in order to remember the sequence and start over.

- Hinepau: Ae, timata.
Awatea: Ko taku wā.
Hinepau: Ah. (Agrees and puts back. They take a few more turns each, and then Toko has a turn out of sequence.)
Hinepau: Haere Toko. (Toko matches a pair.)
Awatea: Eeh, ko a ia (indicating that it was Hinepau’s turn). Ka haere din, din, din (as she is pointing to the sequencing of turns).
Hinepau: (Realising she has missed a turn) Oh Ae, I haere a ia kātahi ko au. Oh, me whakahoki erā (addressing Toko). E tika. Me haere a ia, au, koe (pointing). Terā te take—I haere koe ki terā taha, so me haere koe ki tenei taha X (indicating out-of-turn). So me whakahoki enā e rua.
Toko: All right, ka taea e koe te tiki.
-

- Hinepau: Yes, start.*
Awatea: My turn.
Hinepau: Ah. (Agrees and puts back. They take a few more turns each, and then Toko has a turn out of sequence.)
Hinepau: Start Toko. (Toko matches a pair.)
Awatea: Eeh, hers (indicating that it was Hinepau’s turn). Go (like this) din, din, din (as she is pointing to the sequencing of turns).
Hinepau: (Realising she has missed a turn) Oh yes, he had his turn and then it’s me. Oh, you had better return those (addressing Toko). That’s right. She goes, then me, then you (pointing). It was for that reason that you went to that side, so you had better come to this side, X (indicating out-of-turn). So you had better return those two.
Toko: All right, you can get them.

From Skerrett White (2003), p. 161

The exemplars in *Kei Tua o te Pae / Assessment for Learning* (Ministry of Education, 2004e) provide numerous examples of young children’s emerging metacognitive skills and awareness. In particular, the exemplars in Book 4: *Children Contributing to Their Own Assessment* showcase young children’s ability to set their own goals, assess their own achievements, and take on some of the responsibility for their own learning. The following episode highlights a young child’s developing metacognitive knowledge and awareness of her learning goals in conjunction with her response to making a mistake.

“Oh, No! That’s Not Right!”

The “Oh, no! That’s not right!” exemplar documents Lauren’s developing sense of what is right in terms of spatial orientation within a screen-printing activity. Lauren appears to be developing her own sense of what is right. When she aligns a second template (a basket) over an earlier print (a cat) and makes a second print, she looks at it and says, “Oh, no! That’s not right! The cat needs to be in the basket—not up there!” She tries again, and when she has aligned the basket and cat to her satisfaction and added a few more items to the picture, she comments, “I like that!”

Lauren also appears to be talking herself through the process, self-regulating her learning. The teacher

notes, “As she was drawing, she said, almost to herself, ‘I’ll have to concentrate!’ And she did ...”

The assessment records Lauren’s apparent view that if something is not right, you either redo it or take a creative approach and readjust your goal. She sees mistakes as part of learning. Her response to an unacceptable amount of space in her second attempt to develop the original design was “Never mind, I’ll put some toys in it.” On this occasion, she appears to interpret mistakes as part of the process of completing a task, and she is developing strategies in response to that belief.

From Ministry of Education (2004c), pp. 8–9

With specific reference to early years mathematics, Pappas, Ginsburg, and Jiang (2003) conceptualise metacognition as involving three major components: recognition of mistakes, adaptability or selection of strategies, and awareness and expression of thought. To engage in reflective discussion requires young children to have skills both in verbal communications and in awareness and expression of thought. For instance, when developing early computational understandings, children become aware that the answer to $5 + 2$ can be found by counting on their fingers and they learn to describe this process in a reasonably coherent way. Pappas and colleagues’ analysis of young children’s problem solving revealed few socio-economic status (SES) differences in their metacognitive abilities. Advantaged children in their study were, however, more facile than their less advantaged peers when it came to describing thoughts and explaining ideas. For many young children, their metacognitive development lags behind their informal mathematical competence (Blöte, Klein, & Beishuizen, 2000; Ginsburg, Pappas, & Seo, 2001). For this reason, teachers should not assume that young children who can ably articulate their thinking are necessarily more mathematically competent than their less articulate peers.

Big Math for Little Kids, a US mathematics programme embedded in activities and stories, places emphasis on the development of mathematical and mathematics-related language. Research evaluation (Greenes, Ginsburg, & Balfanz, 2004) provided evidence of young children’s abilities to express their mathematical thinking through discussions involving conjectures, predictions, and verifications.

Which Is Heavier?

An example from the measurement strand demonstrates how children are expected to draw on their everyday knowledge to resolve a contradiction.

Children are shown a picture of an unbalanced seesaw with a frog at one end (the heavier end) and a large bear at the other end (the lighter end). Children are urged to talk about what they see, to identify what is ‘funny’ or ‘wrong with the picture’ and to tell how they know, and later, describe how they would ‘fix it.’ Typical responses demonstrate children’s understanding of heavier/lighter and of balance. For example, one child responded, “The bear is bigger. He’d be at the bottom.” Prompted to speculate about the circumstances under which this could be true, the child answered, “The bear is a balloon. So the frog is heavier!”

From Greenes, Ginsburg, and Balfanz (2004), p. 161

A study by Zur and Gelman (2004) provides evidence that three-year-olds can invoke metacognitive strategies. Using a number game context, the children were able to predict—using predictions that honoured the principles that addition increases numerosity and subtraction decreases numerosity—and then check their predictions by counting. Participation in tasks that involved predicting and then checking by counting not only had the advantage of embedding counting within a meaningful task, it also provided children with the ability to use their counting as a source of authority and sense making, adding to their metacognitive knowledge of themselves as mathematical learners.

Centre–home links

Children’s development of informal mathematical understandings is a result of their natural curiosity and exploration of their environment (Fuson, 1992; Tudge & Doucet, 2004). What they learn during their early years is not, however, learned in isolation. *The Best Evidence Synthesis: The Complexity of Community and Family Influences on Children’s Achievement in New Zealand* highlights the role of both home and community links. Young children’s activities are “complemented by relationships that encourage the gradual involvement of children in the skilled and valued activities of their family and the society in which they live” (Biddulph, Biddulph, & Biddulph, 2003, p. 119). Parents, families/whānau and community members have a critical role to play; they are at the centre of the social contexts in which children live and thus are well placed to provide scaffolding for them as they develop mathematical ideas.

Home-based mathematical practices

The young child’s environment provides a rich source of mathematical experiences. The following vignette illustrates how Freya, a six-month-old baby, initiates an interaction with her mother—an interaction based on a repetitive action related to cause and effect.

Napping or Not?

Freya is seated on her mother’s bed supported by a tripillow. Her mother is to her right. Toys are spread out in front of her. Her toy basket is diagonally opposite. Freya picks up a toy and then mouths it, then moves the object from her right to left hand, looking intently at the object. She leans back into the tripillow and then pulls herself forward. She leans back into the tripillow and looks up at her mother and smiles [mother smiles and kisses her], then pulls herself forward; immediately repeating action in rapid motion. Mother smiles, kisses and hugs baby. Freya repeats this action ten times, always pausing and smiling at her mother [mother responds each time with smiles, kisses and hugs].

Haynes, in interpreting the mathematical experiences present in this episode, suggests that Freya exhibits an awareness of the effect of her actions on producing a guaranteed response. Through the interactions with her mother, Freya is learning about ‘cause and effect’ relationships.

From Haynes (2000a)

In a New Zealand study of young Tongan children, Ha’angana (1999) provides examples of mathematics integrated into the children’s church activities (e.g., holding up a number card and exploring the significance of that number in the Bible, singing number songs). Mathematical experiences in the home were explored in an Australian study (Clarke & Robbins, 2004) that investigated how families from lower socio-economic circumstances conceptualise and enact literacy and numeracy. Parents photographed their child participating in literacy and numeracy activities in their home and community. Photographed activities collected from a total of 52 families included numerations, shape and spatial activities, cooking, game playing, money, sorting and classifying, measuring, and travel and location. From parental discussions of the photographed activities, Clarke and Robbins concluded that “children were engaged in rich and varied mathematical environments and the parents had some awareness of the range of mathematical activities in which their children were engaged” (p. 180). Involvement in the project resulted in increased parental awareness of the scope of numeracy experiences they engage in on a day-to-day basis. Parents were able to make explicit the incidental, though not accidental, numeracy opportunities: “Every time I tuned in he was actually learning, everything was literacy and numeracy” (p. 180).

More detailed studies involving researcher observations of home practices suggest variability in both the quantity and quality of mathematical interactions between adult and child. While there is some evidence that social class differences may be implicated in the extent to which children are involved in mathematical experiences, Tudge and Doucet (2004) claim that the evidence is somewhat uncertain:

For example, Starkey, Klein, and their colleagues (Starkey & Klein, 2000; Starkey et al., 1999) found that middle-class parents reported providing more mathematics activities to their children than did working-class parents, and Saxe and his colleagues (1987) found that middle-class mothers reported that their children engaged in more complex mathematical experiences more often than did working-class mothers. However, Ginsburg and Russell (1981) found no significant variation of performance for a variety of mathematical tasks for children from working- versus middle-class families, and Ginsburg et al. (1998) reported that although “many economically disadvantaged children enter school less than fully prepared to learn formal mathematics” (p. 425) the data provide little evidence that children from different socioeconomic groups have had significantly different mathematical experiences. Low-income mothers, however, tend to believe that preschool teachers are responsible for providing instruction in mathematics (Holloway et al., 1995; Starkey & Klein, 2000). (p. 23)

In their US study, Tudge and Doucet investigated naturally-occurring mathematical activities engaged in by 39 three-year-olds, evenly divided by ethnicity and social class. Each child was observed for 18 hours over the course of a single week in such a way as to cover the equivalent of a complete day in its life, including time in the home, other’s homes, the childcare centre, and public places. Overall, Tudge and Doucet estimated the number of mathematics experiences (mathematical lessons⁸ and mathematical play) that the children from their study had over an entire day to be just ten⁹. The most commonly observed mathematics lesson involved numbering objects. Adults and children were most likely to be engaged in number-based activities and conversations when interacting with puzzles, toys, television, computer programmes, and other games. Likewise, in mathematics-related play activities, children’s most common interaction was via objects (such as toys, puzzles, books, computer programmes) that had number as a central feature.

Although considerable individual variation in the number of mathematical experiences was found, this variation was not explicable by class or by ethnicity. An additional finding was that children were just as likely to be involved in mathematical lessons in the home as in the early childhood centre setting. It was also noted that parents focused more on helping their children learn to read than on their development of mathematical understanding. Given the premise that the activities that routinely take place within the home environment are the key to understanding parents’ cultural construction of their child’s life and development (Harkness & Super, 1995), Tudge and Doucet express concern about the relative lack of importance of mathematics in young children’s lives:

One conclusion that might be drawn is that the provision of mathematical experiences to 3-year-olds is not an important cultural practice, at least with some of the parents whose children we studied. ... there is certainly room for parents and other important people in children’s lives to enhance children’s opportunities for mathematical experiences. (p. 36)

Variation in New Zealand home practices was also a significant finding in the EMI-4s study. Young-Loveridge et al. (1995) found that the mothers of the ‘expert’ children¹⁰ encouraged their children to pursue mathematical activities related to the children’s own interests (e.g., counting the number of engines and carriages on trains, monitoring the speedometer on the car, and dealing with money), as opposed to the interests of other family members. In contrast, although the ‘novice’ children were also involved in a range of mathematical activities at home, their mothers did not comment on incorporating mathematical ideas into activities that were of interest to their children. In an earlier case study of a sample of six five-year-old children,

Young-Loveridge (1989b) found that high scorers on numeracy tasks had:

a wide range of experiences involving numbers, a strong orientation towards numeracy by members of their families, and the opportunity to observe their mothers using numbers to solve everyday problems of their own. The low scorers, on the other hand, had few number experiences, an orientation by their families towards literacy but not numeracy, little opportunity to observe their mother using numbers for the solution of practical problems of their own, as well as relatively low family expectations for their mastery of skills. (p. 43)

The link between the family's 'orientation to number' and the mother's attitudes to mathematics was also an issue in a recent study in England. Aubrey, Bottle, and Godfrey (2003) studied the mathematical development of two 30-month-old children, through analysis of observation and discourse in both home and out-of-home settings. Fine-grained analysis enabled the researchers to posit a relationship between the children's engagement in mathematical experiences and the roles and relationships established with their parents and other significant adults. Quantitative analysis showed that Child L received a fairly constant input of mathematical dialogue over the observation period, whereas Child H received markedly more input with increasing age. Qualitative analysis of the interactions between the mother and child, and parent interviews, pointed to differences in the nature of the interactions and parental input.

The mother of Child H seemed to recognise that talking and communicating were important and saw mathematics as a part of everyday life. Learning for Child H arose from play activities chosen mainly by Child H herself. Her mother did not believe that she should sit down to 'teach' Child H in any formal way, although she did report in interview that she knew other parents did so. The mother of Child L, by contrast, saw mathematics in the home in terms of discrete activities where the goal was the acquisition of counting and arithmetical skills and in which the adult might assume a direct teaching mode. (Aubrey et al., 2003, p. 102)

These two parental styles are in accord with Walkerdine's (1998) typifications of mother-child conversational practice in the home: for the mother of Child H, the focus was the practical accomplishment of a task and the mathematics was incidental; for the mother of Child L, the mathematics was the explicit pedagogical focus of a purposeful activity. The researchers conjecture that the existence of these "distinct parent pedagogical styles of supporting children's early mathematical development" (p. 102), ranging from the didactic to the more genuinely participative, may create a mismatch between what some children experience in their homes and what they experience in more formal early childhood settings.

The widely-accepted features of an appropriate learning environment that is based on a view of 'everyday' mathematics constructed through children's chosen investigations in their social and cultural environment may *not* be one to which all young preschoolers are accustomed. Fostering a positive disposition to learning mathematics where there is opportunity for ideas to be tested out and mistakes to be made will be particularly important in some cases. (Aubrey et al., 2003, p. 103)

As a consequence, they argue, a particular challenge for early childhood educators will be to familiarise young children who are less accustomed to play-oriented approaches and child-directed activity in the home setting with the approaches of the early childhood setting so that positive expectations and attitudes can be acquired for later mathematics learning.

In another investigation of home-based practices, Anderson, Anderson, and Shapiro (2005) observed shared book reading to examine how parents and their young children attended to mathematical concepts. The study involved 39 parents and their four-year-old children, all from a culturally diverse metropolitan area of Canada. Consistent with previous research, the study found that the amount of mathematical talk differed widely across families. During shared reading episodes,¹¹ the illustrations were the most likely stimulus for adult-child interchange. None of the mathematical talk appeared contrived; interactions principally involved attempts

to make meaning or make sense of the story. For example, in the following episode, the mother referenced a specific attribute (curvature) in an attempt to challenge the child's interpretation of the 'bending' tail:

Daughter: He broke his tail.

Mother: His tail is broken? No, I think it's just a curve; it's curving.

Although the research did not measure children's appropriation of mathematical vocabulary, opportunities afforded those children who were exposed to words such as 'bigger', 'small', 'six', 'lots', and 'shape' in the rich context of storybook reading were regarded as beneficial to acquisition of mathematics vocabulary and the associated meanings. Another significant finding from this study was the differential nature of the discourse. Some parents encouraged comparison of the pictorial representations of objects with objects from children's everyday experiences, thereby prototypically modelling the concept of scale in a foundational way. In accord with Heath's (1983) foundational work on literacy practice, the researchers suggested that these parents were laying the groundwork for their children to be able to deal successfully with pictorial representations that they might encounter in mathematics classrooms.

The researchers also noted that the manner in which the families in their study shared mathematics is not entirely consistent with the ways that storybooks might typically be used within a formal learning environment. The tendency of parents to integrate mathematical talk almost seamlessly into storybook reading contrasts with the teacher practice of using a storybook as a springboard to mathematical activities. These observations, they claim, support the notion of a fundamental difference between learning at home and learning in formal settings.¹² While not advocating that teachers try to emulate what parents do or that parents adopt school-like activities at home, Anderson et al. (2005) feel that it is "prudent that educators be aware of and consider these differences" (p. 22).

Facilitating centre–home links

Concerns about differential home experiences in mathematics, combined with concerns about the variability of young children's levels of mathematical understandings (Young-Loveridge, 2004), have led some researchers to trial intervention programmes that focus on home–school links, providing support in terms of communication strategies and home resources. An evaluation of 14 experimental family numeracy programmes in Britain (The Basic Skills Agency, 1998) noted that those programmes found to be most effective had three key strands: joint and separate sessions for parents and children, a structured numeracy curriculum, and bridging activities that parents could use to develop their child's numeracy at home.

Using take-home kits containing materials familiar to the children, Macmillan (2004) describes the development of numeracy concepts among a small group of Aboriginal preschool children in an urban Koori preschool. The numeracy kits, designed to enrich the numeracy concepts of counting, ordering, and pattern, enabled the children's families to support them in their play and develop a better understanding of their children's numeracy potential. Through a process of mutual enculturation, the kits provided a bridge between the families and centre staff.

Including families and whānau in curriculum and assessment activities can also be beneficial in bridging the centre–home community divide. Examples in *Kei Tua o te Pae / Assessment for Learning*, Book 4 (Ministry of Education, 2004a) illustrate how the inclusion of mathematical experiences in construction activities (e.g., Exploring local history, pp. 10–11) and parent or 'whānau' stories and visitor input (e.g., Growing trees, pp. 18–19) can be linked to narrative assessment records. Field trips to local sites of interest can be recorded as wall displays and in the children's 'books' for future community reflection.

While most home–school programmes involve planned action for and by parents, there is recognition of individual caregivers' attempts to develop their own children's numeracy. To

enhance awareness of opportunities to support mathematics exploration and learning, and to mitigate potential negative modelling of mathematics or attitudes (Ginsburg & Golbeck, 2004), parental education programmes are also advocated. The Ministry of Education has in the past funded the Feed the Mind campaign and more recently published supporting material for the parents of children involved in the Numeracy Development Project (Ministry of Education, 2004d). Supporting parents of preschool children through information workshops has also been found to be effective (Griffin & Coles, 1992).

Teacher knowledge and beliefs

Within early childhood education, Hedges (2002) argues that teacher beliefs, rather than teacher knowledge, have traditionally been established as the most important determinant of quality teaching and learning interactions. However, recent moves to embrace a sociocultural perspective have positioned the teacher as having a more active role to play in children's learning, a position that places greater demands of the nature and level of teacher content knowledge. If adults are to fulfil the role of the "knowing assistant and supporter" (Perry & Dockett, 2002, p. 103), they need to understand the mathematics that children are dealing with and to be aware of the many opportunities that present themselves for the learning of mathematics. Studies previously cited in this chapter clearly demonstrate that parents, whānau, and early childhood practitioners who are sensitive to the many learning opportunities that surround them are able to capitalise on those opportunities to extend children's mathematical learning in a wide range of contexts.

However, research studies have also noted that some adults are underutilising the opportunities around them. Siraj-Blatchford et al. (2002) document that, even in the most effective early childhood programmes in Britain, there were examples of teachers having inadequate knowledge and understanding of subject content. While a broad general knowledge was considered to be important, a critical level of specialist subject knowledge was regarded as essential. Content knowledge—mathematical understanding and the associated understanding of children's mathematical development—and teacher confidence as a mathematics learner appear to be at issue (Aubrey, 1997; Davies, 2003; Papic & Mulligan, 2005; Parsonage, 2001; Sarama & Clements, 2004; Young-Loveridge et al., 1998).

One of the tensions for early childhood teaching is that while children demonstrate remarkable facility with many aspects of mathematics, many early childhood teachers do not have a strong mathematical background. At this stage in their development, children's mathematical potential is great and it is imperative that early childhood teachers have the competence and confidence to engage meaningfully with both the children and their mathematics (Perry & Dockett, 2002, p. 107).

Low levels of content knowledge and the resulting lack of confidence about mathematics limit teachers' ability to maximise opportunities for engaging children in the mathematics learning embedded within existing activities, as well as their ability to introduce more focused intervention activities designed to cater for diverse learners. Evidence of restricted mathematics learning occurring in early childhood settings comes from both national and international studies. In the UK, the REPEY study found that approximately 5% of four-year-olds' time was spent doing mathematics activities (Siraj-Blatchford et al., 2002). In New Zealand, Davies (2002) and Young-Loveridge et al. (1995) found that children did not always take advantage of mathematics learning opportunities available in play, particularly with regard to number. Moreover, Davies (2003) found a mismatch between what was claimed in planning and what actually occurred in kindergarten sessions, noting the practitioners were unclear about recognising mathematical learning and expressed low confidence about their own mathematical knowledge.

Professional development implications

Research evidence suggests that professional development that aims to increase teachers' awareness of mathematics in everyday life, to encourage greater awareness of home and cultural factors, and to improve mathematical knowledge and teaching skills provides positive outcomes for both teachers and children. White and Hosoume (1993, cited in Farquhar, 2003) report from a three-year project in the US to improve teachers' mathematics knowledge and teaching skills that improvements in teacher knowledge and confidence are linked to increased use of hands-on activities and greater collaborative engagement. Similarly, in New Zealand, Hedges (2002) noted that as teachers gained confidence in their content knowledge, they were more likely to share this knowledge with children. They were also more likely to be aware of integrated curriculum experiences and opportunities that potentially could strengthen children's mathematical understanding.

Within the initial teacher education sector, Haynes (2000b) demonstrates how the acquisition of knowledge of the primary school curriculum document can enhance the ability of early childhood teachers to provide mathematics learning experiences through play. The early childhood teachers in Haynes' study, enrolled in a dual curriculum programme,¹³ reported that having knowledge of both curricula made them confident in providing children in early childhood settings with a productive start to their mathematics education (Haynes, 2000c).

An unexpected result arose from Clarke and Robbins' (2004) study of parents' interpretation of mathematics in the home. Both the nature of the parents' descriptions and their ability to articulate them came as a surprise to many of the teachers in the project. This is reflected in their comments:

It is interesting because I think they are doing a lot at home, but we are not aware of it under its titles [reference to mathematical concepts such as measurement, space and shape etc.]

(I was surprised) to see that some families were working with children in the kitchen. They pulled up a stool and did counting and measuring.

Breaking from the traditional deficit model, Clarke and Robbins (2004) aimed to change teachers' programmes in ways that supported a diverse range of strategies for numeracy outcomes. Not only did this study demonstrate the extensive numeracy enactments taking place in many families in lower socio-economic circumstances, it also highlighted for teachers the rich constructions of mathematics that individual children were achieving and the important interpersonal relationships that actively supported and extended these constructions. For teachers to cater effectively for children's different and diverse pathways, it is imperative that they connect with and build upon the children's rich base of mathematical experiences in ways that acknowledge and support the family's role.

Bridge to school

Mathematics educators are in accord with early childhood researchers (e.g., Cullen, 1998; Smith, 1996) who argue that learning and development is driven by social interactions in the context of cultural activities. Children's developing understandings are most appropriately situated within "social and cultural contexts that make sense to the children involved" (Perry & Dockett, 2004, p. 103). Thus, to optimise young children's mathematics learning there needs to be sufficient opportunity for them to experience and explore mathematics within everyday experiences, in both informal and formal settings.

However, while many studies demonstrate that increasing children's access to mathematical activities and experiences results in achievement gains, early childhood educators argue that caution is needed to ensure that these gains are not at the expense of personal and social learning. The challenge of an integrated curriculum for young learners is to enhance and highlight mathematical knowledge while supporting an *orientation* towards learning.

The early years curriculum document, *Te Whāriki*, offers a distinctive approach with regard to the history and philosophy of the sector; one that contrasts with what is offered in school curricula. The emphasis in *Te Whāriki* is “a model of learning that weaves together intricate patterns of linked experience and meaning rather than emphasising the acquisition of discrete skills” (Ministry of Education, 1996, p. 41). One of the ways in which *Te Whāriki* sets itself apart is through its outcomes: negotiability and responsiveness to the learner substitute for the specificity and accuracy that is typically associated with more formal mathematics.

Some researchers have expressed concern that these differences may be amplified by recently adopted numeracy practices in early schooling. For example, Carr (1997) cautioned that the test results for the numeracy component of the School Entry Assessment, *Checkout/Rapua*, may become a self-fulfilling prophecy. Specifically, children identified as competent may be given more challenges, while those with low scores may be less likely to be provided with opportunities to develop “dispositions to be courageous, mindful, persistent and responsible” (p. 325). In a similar vein, Peters (2004b) claims that an emphasis on developmental progressions in number may overlook children’s competency in other areas. Because numeracy progressions are just one of many possible ways in which learning can be constructed, Peters (2004b) is concerned that assessment against developmental progressions may “lead to those who do not fit the norm being pathologised, perhaps blaming either the child or family for perceived deficits” (p. 5).

Based on her study of children in transition to school, Peters (2004a) offers a cautionary note about predetermining young children’s capabilities in mathematics. In order to facilitate the transition process, Peters (1998) argues that a more holistic picture of the child is needed. She recommends that it would be good to follow Meisel’s (1992) suggestion and “focus on a child’s current skill accomplishments, knowledge, and life experiences, and then proceed in a differentiated way to extend a child’s mastery to different and more complex levels” (p. 121).

There is considerable disparity in number competency between the most and least competent new entrants (Gilmour, 1998; Young-Loveridge, 1989b; Young-Loveridge, 1993). Diagnostic Interview (NumPA) data from the 2004 *Numeracy Development Project* (see table 3.1) suggest that entry level disparities remain a feature for year 1 students (Young-Loveridge, 2005).

Initial Stage	Percentage of students (rounded to 0.1%)
0: Emergent	15.7
1: One-to-One Counting	29.8
2: Count All with material	43.7
4: Advanced Counting	2.2
5: Early Additive	0.2

Table 3.1. Addition/Subtraction Initial Stages of NumPA for 7793 students

Earlier research studies (e.g. Young-Loveridge, 1991; Wylie, 1998) suggested that children’s numeracy skills may be extremely stable: children who were highly competent at five years maintained their higher levels several years later and the less competent continued to show fewer skills. From her EMI-5s study, Young-Loveridge found that just below 80% of children who began school in the bottom half of their cohort were still in the bottom half of the cohort after four full years at school. Correlations across the data revealed that just over half of the variability in mathematical understanding at nine years could be explained by children’s understanding of number at the age of five.

Recent data from Australia paints a different picture. Analysis by Horne (2005) of the data from a five-year longitudinal study of 572 children from 70 schools involved in the Early Numeracy Research project (ENRP) found that children’s mathematical understanding developed at

different rates and that many moved position relative to their peers. Horne reports that about a third of children who began in the lower part of the class in terms of their mathematical understanding moved into the upper part of the class within five years. This challenges the earlier findings related to stability of competency. Horne claims that:

we need to be very careful not to label children on the basis of demonstrated achievement. Children do learn at different rates ... Students who arrived at school with little knowledge in number domains made considerable gains, often moving ahead of students who had great knowledge. (p. 449)

These results challenge the belief that children who arrive at school in the lower group are condemned to remain in it.

The results also reinforce the fact that it is important for teachers to understand the diverse ways in which a child's development can evolve. Sometimes this development can appear to be 'splintered' or departing from the norm, yet it is still within or close to the range of normal development (Horowitz et al., 2005). Perry and Dockett (2005) argue that the practices embedded in systemic numeracy initiatives in Australia and New Zealand support educators' developing awareness of the knowledge and skills that young children bring to early childhood settings. Unlike findings from earlier studies, in which teachers underestimated children's ability in mathematics and spent much of the first year teaching concepts that the children already knew (Young-Loveridge, 1989a), teachers in these numeracy projects are urged to "become familiar with children's existing mathematical understanding as they commence school to ensure that programming is designed to meet the needs of individual students" (Board of Studies, NSW, 2002, p. 14).

In response to calls for greater harmonisation of approaches to the teaching of mathematics across the pre- and early school years (Hedge, 2003; Macmillan, 2004), researchers and educators note synergies between the mathematics embedded within the curriculum documents. In the following vignette, Perry and Dockett (2005) illustrate how mathematical practices highlighted in the school sector—mathematisation, connections, and argumentation—are also clearly accessible to young children.

Mud Pies

Bruce (4.10) and Giselle (4.11) are in the sand pit near a water trough.

Bruce: [Tips water from a bucket into the trough. The water trough is about one-third full.] "We've got enough water, now let's get back to some more sand."

[Both children shovel sand into the water trough, counting as they go.]

Bruce: "That turns to mud, doesn't it?"

[More shovelling of sand into the trough and counting]

Giselle: "Yeah."

[Boy proceeds to mix the sand and water with his shovel.]

Giselle: "Don't mix it up now. We're still getting sand here. We're still getting sand."

[The children continue adding sand and water until the trough is almost full of a mixture of a consistency appropriate to make 'mud balls'.]

It is evident that these two children play cooperatively to work towards a solution to their problem, using mathematical ideas including early understandings of pattern and ratio. In mathematising their problem they explore the issue of required amounts of sand and water. They provide arguments about the amounts of water and sand needed, and they exhibit number and probability sense using trial and error patterns to test their assumptions about how much sand/water was needed.

From Perry and Dockett (2005)

In a similar manner, Haynes (2000a) provides illuminating examples of how the thinking that underlies mathematical understanding in infants' and young children's play situations aligns with the emergence of concepts as presented in the content strands of the school mathematics curriculum.

A move towards early years (years 0–8) teacher education programmes provides a powerful means to strengthen teachers' understanding of the links between *Te Whāriki* and *Mathematics in the New Zealand Curriculum*. A greater awareness of the activities for the early levels of school learning can help early years educators “recognise and articulate the mathematics children encounter in early childhood, and how this relates to the student of mathematics at the formal and cultural level” (Peter & Jenks, 2000, p. 9). (See also earlier discussion on professional development.)

Collectively, the evidence-based studies presented in this chapter confirm the crucial role that quality early years education plays in the development of infants' and young children's mathematical proficiency. In the following chapters, we continue this journey, looking at how quality teaching in the school sector can best support the mathematics learner.

-
- ¹ The term ‘young learners’ or ‘young children’ is used within this chapter to include ‘infants, toddlers and young children’.
 - ² The five strands in *Te Whāriki* are: Well-being—*Mana Atua*; Belonging—*Mana Whenua*; Contribution—*Mana Tangata*; Communication—*Mana Reo*; and Exploration—*Mana Aotūroa*.
 - ³ www.ioe.ac.uk/playground
 - ⁴ Parents of the target children were interviewed to ensure that the additional mathematical activities matched children's interests.
 - ⁵ www.dfes.gov.uk/research
 - ⁶ Vygotsky (1986) describes the ZPD as the “distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers”.
 - ⁷ Kits containing activities based on Garden Play, Puppet Play, and Pond Play were introduced to provide an additional numeracy focus within three centres.
 - ⁸ Lessons, as contrasted to ‘play with academic objects’ were defined as explicit attempts to impart or elicit information involving competencies in mathematics.
 - ⁹ For a child to be coded as engaged in a mathematical activity, mathematics has to be the focus of attention. For example, laying the table was included if someone pointed out to the children that four people were going to be eating and so he or she would need four forks. The researchers acknowledged that there could well be episodes of mathematical thinking that were not verbalised (e.g., counting silently while walking up steps).
 - ¹⁰ The case study children included four with particularly high levels of numeracy (i.e., experts) and four with particularly low levels of numeracy (i.e., novices).
 - ¹¹ *Mr McMouse and Swimmy*, both authored by L. Lionni.
 - ¹² See also a discussion in [Ainley, J. (1999). Who are you today? Complementary and conflicting roles in school-based research. *For the Learning of Mathematics*, 19(1), 39-47] concerning the contrasting roles of parent/teacher/researcher.
 - ¹³ Specialist degree programmes in New Zealand that focus on Years 0-8.

References

- Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington: Ministry of Education.
- Anderson, A., Anderson, J., & Shapiro, J. (2005). Supporting multiple literacies: Parents' and children's mathematical talk within story book reading. *Mathematics Education Research Journal*, 16(3), 5-26.
- Arakua, B. (2002). *Investigating mathematical interactions in early childhood education*. Unpublished master's thesis, Dunedin College of Education, Dunedin, NZ.

- Aubrey, C. (1997). *Mathematics teaching in the early years: An investigation of teachers' subject knowledge*. London: Falmer Press.
- Aubrey, C., Bottle, G., & Godfrey, R. (2003). Early mathematics in the home and out-of-home contexts. *International Journal of Early Years Education*, 11(2), 91-103.
- Biddulph, F., Biddulph, J., & Biddulph, C. (2003). *The complexity of community and family influences on children's achievement in New Zealand: Best evidence synthesis*. Wellington: Ministry of Education.
- Blöte, A., Klein, A., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction*, 10, 221-247.
- Board of Studies NSW. (2002). *Mathematics K-6: Syllabus*. Sydney: Board of Studies, NSW.
- Carr, M. (1997). *Technological practice in early childhood as a dispositional milieu*. Unpublished D.Phil thesis, University of Waikato, Hamilton.
- Carr, M. (1998). *Assessing children's experiences in early childhood: Part 1*. Waikato: University of Waikato.
- Carr, M., Peters, S., & Young-Loveridge, J. (1991). The informal mathematics of four-year-olds: Understanding its purpose. In M. Gold, L. Foote, & A. Smith (Eds.), *Proceedings of the 5 Early Childhood Convention*. Dunedin: University of Otago
- Carr, M., Peters, S., & Young-Loveridge, J. (1994). Early childhood mathematics: Finding the right level of challenge. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers* (Vol. 1, pp. 271-282). Wellington: Wellington College of Education.
- Casey, B., Kersh, J. E., & Young, J. M. (2004). Storytelling sagas: An effective medium for teaching early childhood mathematics. *Early Childhood Research Quarterly*, 19(1), 167-172.
- Clarke, B., & Robbins, J. (2004). Numeracy enacted: Preschool families' conceptions of their children's engagements with numeracy. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 175-182). Sydney: MERGA.
- Craw, J. (2000). *"Mathematics is a beautiful way of communicating": Using the early childhood teacher's voice*. Unpublished DipMathEd project, University of Auckland.
- Cullen, J. (1988) Preschool children's learning strategies. *SET: Research Information for Teachers*, 2 (Item 15).
- Cullen, J. (1998) Emergent learners: Making the transition of school learning. *Children's Issues*, 2(1), 30-33.
- Davies, N. (1999). *The influences of teacher beliefs on mathematical experiences provided in early childhood centres*. Unpublished research project, Massey University, Palmerston North.
- Davies, N. (2002). *Number games in early childhood centres*. Unpublished master's thesis, Massey University, Palmerston North, New Zealand.
- Davies, N. (2003). Counting on early childhood educators. *Early Education*, 32, 29-35.
- Diezmann, C., & Yelland, N. (2000). Developing mathematical literacy in the early childhood years. In N. Yelland (Ed.), *Promoting meaningful learning: Innovations in education early childhood professionals* (pp. 47-58). Washington, DC: National Association for the Education of Young Children.
- Dockett, S., Perry, B., & Nanlohy, P. (2000). Myths and challenges: Using computers in early childhood services. In S. Schuck & G. Segal (Eds.), *Change and choice in the new century* (pp. 49-55). Sydney: University of Technology.
- Dougherty, B. (2003). Voyaging from theory to practice in teaching and learning: A view from Hawai'i. In N. A. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-31). Honolulu: PME & PMEMA.
- Education Review Office. (2004). *Catering for diversity in early childhood services*. Wellington, NZ: Education Review Office.
- Farquhar, S. (2003). *Quality teaching early foundations: Best evidence synthesis*. Wellington: Ministry of Education.
- Fleer, M. (2002). Sociocultural assessment in early years education: Myth or reality. *International Journal of Early Years Education*, 10, 105-120.
- Fuson, K. (1992). *Children's counting and concepts of number*. New York: Springer-Verlag.
- Gifford, S. (2004). A new mathematics pedagogy for the early years: In search of principles for practice. *International Journal of Early Years Education*, 12(2), 99-115.
- Gifford, S. (2005). *Teaching mathematics 3 - 5*. Maidenhead: Open University.
- Gilmore, A. (1998). *School entry assessment: The first national picture- July 1997-May 1998*. Christchurch: Unit for Studies in Educational Evaluation, University of Canterbury.
- Ginsburg, H., & Golbeck, S. (2004). Thoughts on the future of research on mathematics and science learning and education. *Early Childhood Research Quarterly*, 19, 190-200.
- Ginsburg, H., Klein, A., & Starkey, P. (1998). The development of children's mathematical teaching: Connecting research with practice. In I. Sigel, K. Renninger (Eds.), *Handbook of child psychology: Vol. 4. Child psychology in practice* (pp. 401-476). New York: Wiley.
- Ginsburg, H., Pappas, S., & Seo, K. (2001). Everyday mathematical knowledge: Asking young children what is developmentally appropriate. In S. Golbeck (Ed.), *Psychological perspective on early childhood education: Reframing dilemmas in*

- research and practice* (pp. 181-219). Mahwah: Lawrence Erlbaum Associates.
- Greenes, C., Ginsburg, H. P., & Balfanz, R. (2004). Big Math for Little Kids. *Early Childhood Research Quarterly*, 19(1), 159-166.
- Griffin, R., & Coles, J. (1992). Assisting parents to develop their preschool children's mathematics. *SAMEpapers*, 81-89.
- Griffin, S. (2004). Building number sense with Number Worlds: a mathematics program for young children. *Early Childhood Research Quarterly*, 19(1), 173-180.
- Ha'angana, E. (1999). *Poto'i Fika: Early childhood mathematics in Tongan children*. Unpublished master's thesis, Auckland University, Auckland, New Zealand.
- Harkness, S., & Super, C. (1995). *Culture and parenting*. In M. Bornstein (Ed.), *Handbook of parenting: Biology and ecology of parenting* (pp. 211-234). Mahwah: Erlbaum.
- Haynes, M. (2000a). The mathematical world of the infant and toddler. *The First Years: New Zealand Journal of Infant and Toddler Education*, 2(1), 16-20.
- Haynes, M. (2000b). Mathematics education for early childhood: A partnership of two curriculums. *Mathematics Teacher Education and Development*, 2, 93-104.
- Haynes, M. (2000c). Teacher education for early childhood through the New Zealand Curriculum Framework. *New Zealand Research in Early Childhood Education*, 3, 163-176
- Heath, S. (1983). *Ways with words: Language, life and work in communities and classroom*. Cambridge: Cambridge University Press.
- Hedges, H. (2002). *Subject content knowledge in early childhood curriculum and pedagogy*. Unpublished master's thesis, Massey University, Palmerston North, NZ.
- Hedges, H. (2003). A response to criticism and challenge: Early literacy and numeracy in Aotearoa/New Zealand. *New Zealand Research in Early Childhood Education*, 6, 13-20.
- Hill, D. (1995). *MinZC-An empowering framework for early childhood mathematics*. Presentation at New Zealand Association of Mathematics Teachers biennial conference, Auckland.
- Horne, M. (2005). The effects of number knowledge at school entry on subsequent number development: A five-year longitudinal study. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the annual conference of the Mathematics Education Research Group of Australasia, pp. 443-450). Sydney: MERGA.
- Horowitz, F., Darling-Hammond, L., & Bransford, J. (2005). Educating teachers for developmentally appropriate practice. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world* (pp. 88- 125). San Francisco: Jossey-Bass.
- Hughes, M. (1996). *Children and number: Difficulties in learning mathematics*. Oxford: Blackwell.
- Hunting, R., & Pearn, C. (2003). The mathematical thinking of young children: Pre K-2. In N. A. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 187). Honolulu: PME & PMEMA.
- Irwin, K. & Ginsburg, H. (2001). Early mathematical discourse. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 185-192). Utrecht: PME.
- Jordan, B. (2003). *Professional development making a difference for children: Co-constructing understandings in early childhood centres*. Unpublished doctoral dissertation, Massey University, Palmerston North, New Zealand.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Macmillan, A. (2002). Numeracy play: How mathematical is it? *Australian Primary Mathematics Classroom*, 7(4), 9-15.
- Macmillan, A. (2004). Take-home numeracy kits for preschool children. *Australian Journal of Early Childhood*, 29(2), 29-39.
- McNaughton, S. (2002). *Meeting of minds*. Wellington: Learning media
- Ministry of Education. (1996). *Te Whāriki. He Whāriki Mātauranga mō ngā Mokopuna o Aotearoa: Early childhood curriculum*. Wellington: Learning Media.
- Ministry of Education. (2004a). *Assessment and learning / Community te aromatawai me te Ako: Hāpori*. Wellington: Learning Media.
- Ministry of Education. (2004b). *Assessment for infants and toddlers: He aromatawai kōhungahunga, tamariki*. Wellington: Learning Media.
- Ministry of Education. (2004c). *Children contributing to their own assessment: Ngā huanga tamariki ki tō rātou aromatawai*. Wellington: Learning Media.
- Ministry of Education (2004d). *Help your child to develop numeracy ... What you do counts!* Wellington: Learning Media
- Ministry of Education. (2004e). *Kei tua o te pae / Assessment for learning: Early childhood exemplars*. Wellington: Learning Media.
- Ministry of Education (2007). *Kei tua o te pae / Assessment for learning: Early childhood exemplars book 17*. Wellington: Learning Media.

- Ministry of Education (in press). *Early Childhood Mathematics Exemplars*. Wellington: Learning Media.
- Papic, M., & Mulligan, J. (2005). Preschoolers' mathematical patterning. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of Mathematics Education Research Group of Australasia, pp. 609-616). Sydney: MERGA.
- Pappas, S., Ginsburg, H., & Jiang, M. (2003). SES differences in young children's metacognition in the context of mathematical problem solving. *Cognitive Development, 18*, 431-450.
- Parsonage, N. (2001). Teacher knowledge: A crucial factor in supporting mathematical learning through play. *ACE papers, 11*, 51-61.
- Perkins, M. (2003). Mathematical babies. *The First Years: New Zealand Journal of Infant and Toddler Education, 5*(2), 11-28.
- Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 81-111). London: Lawrence Erlbaum Associates.
- Perry, B., & Dockett, S. (2004). Mathematics in early childhood education. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia* (pp. 103-125). Flaxton: Post Pressed.
- Perry, B., & Dockett, S. (2005). What did you do in maths today? *Australian Journal of Early Childhood, 30*(3), 32-37.
- Peters, S. (1992). Spatial ability: An investigation into the spatial ability and play preferences of four-year-old children. *SAMEpapers, 117-129*.
- Peters, S. (1998). Gathering and sharing information on young children within the context of the transition to school. *SAMEpapers, 107-122*.
- Peters, S. (2004a). "Crossing the border": An interpretive study of children making the transition to school. Unpublished PhD, University of Waikato, Hamilton.
- Peters, S. (2004b). *Making the links between early childhood mathematics and school mathematics: An Aotearoa/New Zealand perspective*. Paper presented at the International Congress for Mathematics Education-10, Copenhagen.
- Peters, S., & Jenks, J. (2000). *Young children's mathematics: A supporting document for the "Making things count" resource*. Wellington: Victoria University of Wellington.
- Pound, L. (1999). *Supporting mathematical development in the early years*. Philadelphia: Open University Press.
- Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York, Oxford University Press.
- Royal Tangaere, A. (1997). *Learning Māori together: Kohanga reo and home*. Wellington: New Zealand Council for Educational Research.
- Sarama, J., & Clements, D. (2004). Building blocks for early childhood mathematics. *Early Childhood Research Quarterly, 19*, 181-189.
- Schwartz, S. (2005). *Teaching young children mathematics*. Westport, CT: Praeger Publishers.
- Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups can see as different? - Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237-309.
- Siraj-Blatchford, I., & Sylva, K. (2004). Researching pedagogy in English pre-schools. *British Education Research Journal, 30*(5), 713-730.
- Siraj-Blatchford, I., Sylva, K., Muttock, S., Gilden, R., & Bell, D. (2002). *Researching effective pedagogy in the early years*. London: Institute of Education, University of London/Department for Education and Employment.
- Skerrett White, M. (2003). *Kia mate rā anō a tama-nui-te-rā: Reversing language shift in Kōhanga Reo*. Unpublished doctoral dissertation, University of Waikato, Hamilton, New Zealand.
- Smith, A. (1996). The early childhood curriculum from a sociocultural perspective. *Early Child Development and Care, 115*, 51-64.
- Smith, A. (1999). Quality childcare and joint attention. *International Journal of Early Years Education, 7*(1), 85-98.
- Sophian, C. (2004). Mathematics for the future: Developing a Head Start curriculum to support mathematics learning. *Early Childhood Research Quarterly, 19*(1), 59-81.
- Starkey, P., Klein, A., & Wakeley, A. (2004). Enhancing young children's mathematical knowledge through a pre-kindergarten mathematics intervention. *Early Childhood Research Quarterly, 19*(1), 99-120.
- The Basic Skills Agency. (1998). *Family numeracy adds up: Lessons from the Family Numeracy Pilot Programme*. London: The Basic Skills Agency
- Tudge, J., & Doucet, F. (2004). Early mathematical experiences: observing young Black and White children's everyday activities. *Early Childhood Research Quarterly, 19*(1), 21-39.
- Vygotsky, L. (1986). *Thought and language*. Cambridge: MIT Press.
- Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.
- Watego, L. (2005). An indigenous perspective on mathematics contextualisation in a pre-school: From safety to empowerment. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 769-774). Sydney: MERGA.

- Worthington, M., & Carruthers, E. (2003). *Children's mathematics: Making marks, making meaning*. London: Paul Chapman Publishing.
- Wylie, C., & Thompson, J. (1998). *Competent children at 6: Families, early education and schools*. Wellington: New Zealand Council for Educational Research.
- Yelland, N. (2005). Curriculum, pedagogies and practice with ICT in the information age. In N. Yelland (Ed.), *Critical issues in early childhood education* (pp. 224-242). Maidenhead: Open University.
- Young-Loveridge, J. (1989a). The development of children's numbers concepts: The first year of school. *New Zealand Journal of Education Studies*, 24(1), 47-64.
- Young-Loveridge, J. (1989b). The relationship between children's home experience and their mathematical skills on entry to school. *Early Childhood Development and Care*, 43, 43-59.
- Young-Loveridge, J. (1991). *The development of children's number concepts from ages five to nine* (Vol. 1). Hamilton: University of Waikato.
- Young-Loveridge, J. (1993). *The effects of early intervention. The EMI-5s Study*. Volume 1: Report of Findings. Hamilton: University of Waikato.
- Young-Loveridge, J. (2004). Effects on early numeracy of a program using number books and games. *Early Childhood Research Quarterly*, 19(1), 82-98.
- Young-Loveridge, J. (2005). Patterns of performance and progress: Analysis of 2004 data. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 5-20). Wellington: Ministry of Education.
- Young-Loveridge, J., Carr, M., & Peters, S. (1995). *Enhancing the mathematics of four-year-olds: The EMI-4s study*: School of Education, University of Waikato.
- Young-Loveridge, J., Peters, S., & Carr, M. (1998). *Enhancing the mathematics of four-year-olds: An overview of the EMI-4S Study*. *Australian Research in Early Childhood Education*, 1, 82-93.
- Zur, O., & Gelman, R. (2004). Young children can add and subtract by predicting and checking. *Early Childhood Research Quarterly*, 19(1), 121-137.

4. *Mathematical Communities of Practice*

This chapter is all about the way teachers arrange for learning within their classroom communities. It provides evidence of the sort of pedagogical arrangements that contribute to positive outcomes for diverse students. In it, we explore how teachers work at establishing a web of relationships within the classroom community with a view to developing their students' mathematical competencies and identities. It is in the classroom community that students develop the sense of belonging that is essential if they are to engage with mathematics. It is within this community that the teacher creates a space for individual thinking and for collaborative mathematical explorations.

Of course, creating a caring learning environment for individuals and groups is not the only thing that effective teachers do in their classrooms. What they do is very much dependent, in the first place, on sound content knowledge. This chapter provides convincing evidence that teacher knowledge is a prerequisite for accessing and assessing students' thinking. It is also critical if teachers are to provide responsive support directed towards structuring and advancing their students' thinking. We look at the ways in which effective teachers scaffold learning through the development of classroom norms of participation, and we see how these are used as a starting point for developing productive mathematical experiences.

Enhancing academic and social outcomes is the objective of all quality teaching. We acknowledge the importance of the teacher's explicit instruction in achieving this objective. Language plays a key part in shaping students' mathematical experiences. Teachers who use mathematical language effectively use it for cognitive structuring. They also use student explanation and justification as a basis for their on-the-spot decision making about the best ways to advance student understanding.

Mathematics teaching for diverse learners demands an ethic of care

Teachers who create effective classroom communities care about student engagement. They demonstrate their caring in their relations with their students (Noddings, 1995), establishing a classroom space that is hospitable as well as 'charged' (Palmer, 1998). They do more than comply with "the politeness norm that dominates most current teacher discourse" (Ball & Cohen, 1999, p. 27). Teaching practice founded on an ethic of care takes pains to ensure that students do not develop a permanent dependency on their teachers. Instead, it works at developing interrelationships that open up spaces for students to develop their own mathematical identities, providing them with opportunities to ask why the class is doing certain things and with what effect (Noddings, 1995). Teachers who care work hard to find out what helps and what hinders students' learning (Cobb & Hodge, 2002). Caring relationships are oriented towards enhancing students' capacity to think, reason, communicate, reflect upon and critique what they do and say in class. Such relationships always involve reciprocity and a pedagogical attention that moves students towards independence (Hackenberg, 2005).

There is a substantial body of literature that reveals that teachers who care are those who identify, recognise, respect, and value the mathematics of diverse cultural groups (e.g., Goos, 2004). Caring about students from diverse cultural backgrounds requires teachers to move closer to their students, which carries with it the implication that teachers also have something to learn from their students (Perso, 2003). Mathematics teaching in kura kaupapa embraces the concepts, practices, and beliefs of te ao Māori (the Māori world) and is deeply committed to Māori culture and language (Macfarlane, 2004). Teachers in kura kaupapa Māori hold that "affectionate nurturing breeds happy hearts and lighthearted spirits and, thereby, warm and caring people" (Ministry of Education, 2000, p. 23). They maintain that such nurturing in a caring learning environment will contribute to positive lifelong futures. Mutual responsibilities

are created in a caring, supportive environment as older children care for younger ones and assist in their learning activities.

Research by Stipek, Salmon, Givvin, Kazemi, Saxe, and MacGyvers (1998) found that “a positive affective climate that promoted risk-taking was positively associated with students’ mastery orientation, help-seeking and positive emotions associated with learning fractions” (p. 483). In their comprehensive study of 24 teachers’ practice in grades 4–6, the researchers compared the practices of non-traditional and traditional teachers. The teachers were videotaped for two or more episodes as they taught the same content. Three teacher practice categories were created from an analysis of the videotaped lessons: (a) learning orientation, (b) positive affect, and (c) differential student treatment. The following vignette illustrates what teachers in the research did to contribute to students’ cognitive and emotional development.

Promoting Student Development

In the study undertaken by Stipek and colleagues (1998) teachers who scored high on the Learning Orientation subscale conveyed to students that effort and persistence would pay off. In whole-class settings this orientation was demonstrated by the teacher staying with one student for a substantial length of time in an effort to get a clearer explanation, to provide alternative suggestions, or, in general, to make sure that the student understood the concept or problem. During student-work time this orientation was observed when the teacher encouraged students to keep working or thinking about a problem, gave them instrumental help that facilitated their progress, allowed plenty of time for students to complete their work, required students to go back and try again when they had reached inadequate solutions, or encouraged them to come up with multiple strategies.

Teachers who scored high on this subscale neither embarrassed students nor ignored wrong answers in whole-class instruction. Rather, they used students’ inadequate solutions and mistakes to enhance the instruction. They commented on the problem-solving process or the strategies students were employing, often making reference to the particular mathematical concepts that students were learning, and they held students to high standards, asking them to explain their thinking in writing as well as verbally.

Teachers with high scores on the Learning Orientation scale also fostered student autonomy by pointing out resources in the classroom, encouraging students to engage in self-evaluation, encouraging and accepting students’ own strategies for solving problems, and giving students choices in how they solved their problems or showed their work. They encouraged students to monitor their own work, set goals, plan their approaches, and move on to the next task without having to check with the teacher. Teachers scoring high on the Learning Orientation scale did not emphasise performance (e.g., receiving good grades), and they did not encourage students to avoid difficult tasks (e.g., by saying, “You’re not ready for those yet”).

Teachers who scored high on the Positive Affect subscale were sensitive and kind (without being artificially sweet). They showed an interest in what students had to say, listened to their ideas, avoided sarcasm or put-downs, and did not allow students to put each other down. The teachers appeared genuinely to like and respect their students. They also appeared to enjoy mathematics, and they made an effort to make mathematics problems interesting. The teachers conveyed that they valued all students’ contributions by, for example, calling on students having difficulties and pointing out what could be learned from mistakes. They never threatened students with “being called on” (and potentially embarrassed) as a means of ensuring their attention.

Teachers who scored high on both Learning Orientation and Positive Affect created an environment that promoted risk-taking which, in turn, contributed to students’ cognitive and emotional development.

From Stipek et al. (1998)

Relationship building

The development of self and others, principled upon an ethic of care, is not always evident in mathematics classrooms. Many Pākehā classrooms tend to undervalue the *whakawhanau-ngatanga*, or relationship building, that supports a caring and knowledge-producing community (Ballard, 2003; Bishop & Glynn, 1999). Bishop and Glynn speak of “a pattern of dominance and subordination and its constituent classroom interaction patterns (pedagogy) that perpetuates the non-participation of many young Māori people in the benefits that the education system has to offer” (p. 131). This pattern often emerges alongside a teacher’s low expectations for some students, relative to the expectations held for others. Jones (1986) quotes a teacher at an all-girls secondary school:

Some of these [Pacific Island] girls have expectations way too high. We get employers ringing up for shop assistants and they [students] don’t want the job! It’s very hard to get them to lower their sights. They shouldn’t have been so high on the first place ...” (p. 489)

Higgins (2005) looked specifically at the practices, and the beliefs underpinning those practices, of an effective mathematics teacher in an English-medium primary school classroom of mainly Māori students. In this classroom, the teacher used the metaphor of the *waka* to describe the class; the members of the class were all heading in the same direction, but different students had different talents and were able to do different things. The teacher also used the *koru* as a metaphor—to describe how the students were growing mathematically, were emerging in different ways, and were helping others to emerge more successfully than would be the case if they were learning alone.

Angier and Povey (1999) investigated the culture within one year 10 mathematics classroom. They provide evidence that students’ academic and social outcomes were greatly enhanced by the inclusive pedagogy of mathematics that this teacher had established. This was a culture that did not minimise individuals’ experiences and contributions within the classroom; nor were collective experiences downplayed. Participation in this classroom went hand in hand with students’ responsibility for themselves and for their own learning. The teacher provided students with opportunities to exercise that responsibility judiciously with respect to one another and to her. Students in the class commented:

She treats you as though you are like ... not just a kid. If you say look this is wrong she’ll listen to you. If you challenge her she will try and see it your way. (p. 157)

She doesn’t regard herself as higher. (p. 157)

She’s not bothered about being proven wrong. Most teachers hate being wrong ... being proven wrong by students. (p. 157)

It’s more like a discussion ... you can give answers and say what you think. (p. 157)

We all felt like a family in maths. Does that make sense? Even if we weren’t always sending out brotherly/sisterly vibes. Well we got used to each other ... so we all worked ... We all knew how to work with each other ... it was a big group ... more like a neighbourhood with loads of different houses. (p. 153)

Valuing students’ contributions

The value that is given to their thinking and their contributions influences the way in which students view their relationship with mathematics. Whitenack, Knipping, and Kim (2001) document how a teacher communicated the value of student effort and knowledge generated in individual, paired or whole-class activity. By validating contributions and asking further questions with the intent of allowing other students to access knowledge, the teacher used students’ ideas to shape instruction and to occasion particular mathematical understanding in the classroom. Bartholomew (2003), however, found that teachers do not always value students’

contributions equally. She found that mathematics teachers at a London school valued the experiences and contributions of top-stream students more highly than the experiences of other students. This evaluation was communicated to students in a range of subtle ways. With his low-stream class, a teacher was authoritarian in manner, “insisting that students queue outside the room in absolute silence and eventually counting them in and seating them alphabetically. They had to remain in their seats in silence, were given no opportunities to ask questions, with the result that many students were extremely confused as to what they were meant to be doing” (p. 131). By contrast, the teacher’s interactions with his top-stream class at the same year level were friendly and jocular. Within this culture, many boys appeared to thrive but most girls did not. Bartholomew found that whilst the girls were performing academically just as well as the boys, they perceived themselves as struggling with the work. Others in the class, too, came to believe in the girls’ poorer ability.

In their New Zealand Progress at School study, Nash and Harker (2002) illustrate how profoundly inequitable instructional attention can affect students. They found that teachers who distribute their attention differentially tend to offer less encouragement to students that they have stereotyped as ‘not mathematical’. One student in their study said: “Like when you ask the teachers you think, you feel like you don’t know, you’re dumb. So it stops you from asking the teachers, yeah, so you just try to hide back, don’t worry about it. Everyday you don’t understand, you just don’t want to tell the teacher” (p. 180). The same inclination to hold back from asking the teacher was expressed by secondary school students in a study by Anthony (1996):

Karen: I honestly thought it was called a pictograph. I don’t want to say anything in case it is so far wrong I embarrass myself.

Lucy: You feel a bit dumb asking questions. I sometimes ask, but if I got one wrong and the rest right I wouldn’t really worry.

Jane: Some of the time I don’t understand the stuff enough in mathematics to answer questions ’cause I’ll probably get it wrong. I only answer questions if I know the answers.

Brooks and Brooks (1993) have observed that students’ unwillingness to answer a teacher’s questions (unless they are confident that they already know the sought-after response) is a direct consequence of the teacher’s use of questions. “When asking students questions, most teachers seek not to enable students to think through intricate issues, but to discover whether students know the ‘right’ answers” (p. 7). The caring teacher, on the other hand, uses power and influence legitimately (Bishop, 1988) to construct more equitable relationships through the discursive practices of the classroom. Hoyles (1982, p. 353) provides a classroom example from her research in which power relations are less than benevolent:

P: ... the teacher was always picking on me.

I: Picking on you?

P: Yes, and in one lesson she jumped on me; I wasn’t doing anything but she said come to the board and do this sum fractions it was. My mind went blank. Couldn’t do nothing, couldn’t even begin.

I: What did you feel then?

P: Awful, shown up. All my mates was laughing at me and calling out. I was stuck there. They thought it was great fun. I felt so stupid I wanted the floor to open up and swallow me. It was easy you know. The teacher kept me there and kept on asking me questions in front of the rest. I just got worse. I can remember sweating all over.

Boaler, William, and Brown (2000) traced the effects on students’ perceptions of mathematics as they moved from a year 8 to a year 9 class. Using questionnaires given to 943 students, interviews with 72 students, and about 120 hours of observations in classrooms, they found

that students in lower streamed classes had fewer instructional opportunities to learn. Teachers ignored their backgrounds and needs and addressed them in ways that exacerbated their difference from more able students. Instructional strategies in these low-stream classes were narrowly defined, resulting in profound and largely negative learning experiences. One secondary school student reflected on the previous year's experience: "... in my primary school we weren't in groups for like how good you are in certain subjects. We were just in one massive group, we did everything together. You've got some smart people and you've got some dumb people in the class, so you just blend in, sort of so you don't *have* to be that good and you don't *have* to be that bad" (p. 645).

Social nurturing and confidence building

"Caring and support is integrated into pedagogy and evident in the practice of teachers and students" (Alton-Lee, 2003, p. 89). Caring within the community of learners is fuelled by the values that underwrite classroom events. An increasing body of literature within mathematics education explains that the values espoused and taught in the classroom play a central role in establishing a sense of social and mathematical identity for mathematics students and significantly shape students' level of engagement with mathematical activity. FitzSimons, Seah, Bishop, and Clarkson (2001), in their *Values and Mathematics Project* (VAMP), investigated the kinds of values that teachers in primary and secondary classrooms espoused and taught. The teachers maintained that they encouraged clarity, flexibility, consistency, open-mindedness, persistence, accuracy, efficient working, systematic working, enjoyment, effective organisation, creativity, and conjecturing. However, the researchers found that, on the whole, the values the teachers claimed they were implementing were actually not the ones witnessed in their classrooms. This finding, and other similar findings (e.g., Bana & Walshaw, 2003; Lin & Cooney, 2001), caution us that the way in which teachers describe their work does not always correspond to the way in which observers characterise that same practice. These findings point to the need to either substantiate teacher assertions or look beyond the 'espoused' forms of practice for evidence.

Social nurturing and confidence building are among the many important aspects of teaching. In a study by Waxman and Zelman (cited in Bishop, Hart, Lerman, & Nunes, 1993), one preservice teacher talked about the way in which her mathematics confidence was destroyed:

"... I hate maths ... I recall heading a page of my maths book 'funny sums'. For this I was sent to stand outside the class, told I was a 'bloody imbecile' and that I was the worst student he had ever had the misfortune to teach ..." (p. 27)

By way of contrast, we provide the following vignette of a teacher focused on social nurturing and confidence building. Students in a classroom study undertaken by Cobb, Perlwitz, and Underwood-Gregg (1998) had been investigating the problem: "How many runners altogether? There are six runners on each team. There are two teams in the race."

How Many Runners?

- Teacher: Jack, what answer – solution did you come up with?
- Jack: Fourteen.
- Teacher: Fourteen. How did you get that answer?
- Jack: Because 6 plus 6 is 12. Two runners on two teams ... (Jack stops talking, puts his hands to the sides of his face and looks down at the floor. Then he looks at the teacher and then at his partner, Ann. He turns and faces the front of the room with his back to the teacher and mumbles inaudibly.)
- Teacher: Would you say that again. I didn't quite get the whole thing. You had – say it again please.
- Jack: (Softly, still facing the front of the room.) It's six runners on each team.

Teacher: Right.
Jack: (Turns to look at the teacher.) I made a mistake. It's wrong. It should be 12.
(He turns and faces the front of the room.)

Once he realised that his answer was incorrect, Jack interpreted the situation as one that warranted acute embarrassment.

Teacher: (Softly.) Oh, okay. Is it okay to make a mistake? Andrew?
Andrew: Yes.
Teacher: Is it okay to make a mistake, Jack?
Jack: Yes.
Teacher: You bet it is. As long as you're in my class it is okay to make a mistake. Because I make them all the time, and we learn from our mistakes, a lot. Jack already figured out, "Oops, I didn't have the right answer the first time," (Jack turns and looks at the teacher and smiles) but he kept working at it and he got it.

From Cobb, Perlwitz, and Underwood-Gregg (1998)

Social nurturing and confidence building was also investigated by Morrone, Harkness, D'Ambrosio, and Caulfield (2004). In their study of middle school students, they revealed that a caring culture within the classroom community contributed to the development of mathematical as well as rounded social identities. Morrone and colleagues found that care within the community of practice was related to an overall goal structure that included consistent affective support. This support conveyed the message that student ideas were valued. In turn, the positive support from teachers encouraged further student effort.

In the New Zealand context, Hyett (2005) explored how teachers created a caring community for diverse students in year 11 mathematics classrooms. One of the three teachers in the study noted: "To me it's all about building relationships with my students ... with them having a positive approach to the subject then they develop the will and the desire and maybe the need at times to do what is required, not to please me but because they realise how far it gets them" (p. 23). Friendliness also means maintaining professional distance:

You can have a friendly relationship with them but at all times I am the teacher. And I'm quite strict about that too, that for my own self, I am a role model; I am a teacher. I need to earn their respect. It's a two-way process. You have to model the behaviour that you want the kids to do ... I don't have many rules in my classroom. The basic one of course is respect for me and other students and that if I'm talking then they're not. (p. 24)

Christensen (2004) also provides evidence of teachers who have simultaneously developed students' social and cognitive development. Te Poutama Tau teachers, working in an intensive professional development programme for Māori-medium classroom teachers, have improved students' achievement in numeracy as they have been responsive to the goal of Māori language revitalisation. Like the English-medium Numeracy Development Project (NDP), the project is part of the Ministry of Education's Numeracy Strategy, where the primary objective is to improve student achievement in numeracy through lifting teachers' professional capability. Student engagement through active participation in mathematical discussion is advocated in Te Poutama Tau (Trinick, 2005). Teachers involved in the project are encouraged to assess the ways in which the social norms of classroom interaction influence student engagement levels.

In 2005 Trinick looked closely at the effects that the project had had on students. He reported a significant change over the duration of the project in terms of student attitude to Mathematics/Pāngarau. In particular, the students expressed a greater comfort level, an increased knowledge base, and higher confidence as a result of the way in which numeracy was taught. There are a number of possible explanations for this improvement. It can be explained by teachers' enhanced mathematical understanding and resulting confidence, following their involvement

in the project. It might also be attributed to the structure that the project provides: the model for assessing students' strategy and knowledge levels, and the suggested learning experiences and equipment that correspond to the different levels of development. Irwin (2005) reported similar findings from the Effective Numeracy Teaching in Years 1 to 6 (Pasifika Focus) project. This project involved 35 teachers from three schools and consisted of extra workshops for teachers and a redefined role for the numeracy leader or facilitator. This redefinition included allowance for additional teacher observations to be made at teachers' request. The teachers involved reported a new empowerment to make changes in their practices.

Barton, Paterson, Kensington-Miller, and Bartholomew (2005) offer preliminary support for enhanced participation and achievement in mathematics from predominantly Māori, Pasifika and new immigrant senior secondary school students. The students involved in the five-year project are from eight schools in a low socio-economic area of Auckland. The Mathematics Enhancement Project (MEP) is driven by a care and concern about what mathematics can do for students who are economically and materially disadvantaged. Over 80% of the students involved are bilingual, and more than half of these understand at least three languages. The project works on four levels: teacher development, learning support for students, image enhancement of mathematics, and classroom-based research. "Coming to these sessions in groups, it's made me look at what my students feel when they're in the classroom. And I think initially ... when somebody comes up with something, you're taken aback a bit ... Because anything that's unknown is scary" (p. 83). The episode signals the crucial importance of establishing a caring community of practice, both for students and their teachers.

Caring about the development of mathematical proficiency

In her landmark study with low-attaining students, Watson (2002) found that teachers who care promote mathematical thinking and reasoning. Specifically, teachers in the study believed that students want to learn in a 'togetherness' environment, that students' questions should propel teaching and learning, and that teaching should foster an awareness of learning. They believed that teaching should not offer students simplified tasks but should challenge them and provide support for them to task risks. The contrasting deficit model of student ability is, however, evident in some schools. For example, the eight teachers in Bergqvist's (2005) research tended to underestimate their students' reasoning ability: they believed that only a few students in a class were able to use higher level reasoning in mathematics. This underestimation, it was argued, may have strong consequences for students' learning in classrooms. Hiebert et al. (1997) discuss a teacher who did not underestimate the mathematical reasoning powers of a student with Down's syndrome. Just like the others in the group, this student took her turn to take on the role of teacher by sharing her word problem and asking questions. Although the problem was relatively easy, the group responded to the student with the same respect and intellectual support they had shown to other members of the group. One student, new to the group, who had called out "that's easy!" was summarily put in her place by another who pointed out that it was not easy for this particular student. The teacher had surrounded the group with the resources of a caring community, intent on promoting the mathematical advancement of all members within the group.

Horowitz, Darling-Hammond, and Bransford (2005) offer the following vignette that describes what a first-year teacher did to organise a supportive, enriching environment for young mathematicians.

The Reams of Paper Problem

On a spring morning just before the last week of school, when many students are just bidding time, Jean Jahr's classroom of 28 second- and third-grade students is intently engaged in a mathematical investigation. A first-year teacher, Jean teaches at an elementary school. The multiracial, multilingual class of students is working in small groups on a single problem. Some children use calculators; others do not. Some have drawn clusters of numbers; others have developed a graphic display for their problem. As they finish, everyone takes their solutions with them as they sit on the carpeted meeting area facing the board. Jean begins by reading the problem with the group: In September, each person in classroom 113 brought one ream of paper. There are 500 sheets of paper in one ream. There are twenty-eight children in class 113. How many pieces of paper were there altogether?

She opens the discussion with an invitation, "Let's talk about how different people solved the problem, and why you decided to solve it that way." Over the next twenty minutes, students show, draw, and discuss seven different strategies they have used to solve the problem. Jane questions them to draw out details about their solution strategies and frequently recaps what students say. With patience and careful choice of words, she helps each member of the group understand the thought processes of the others. As the session nears its end, she asks if everyone understands the different solutions. Three children from one group seem in doubt and raise their hands. Jean asks one of the girls to come up and show "her way." The teacher and the other children observe patiently, obviously pondering the girls' thought process. Suddenly, Jean's face lights up as she sees what they have done. Her response clarifies their work: "That's how you did it! I was wondering if you had used tens groupings, but you had a totally different pattern. You started as if there were 30 children and then you subtracted the 1,000 sheets that would have been brought by the additional two children from the total number. You rounded to a higher number and then you subtracted. Wow. I get it. Let me see if I can show it to the others."

The young girl is pleased when the teacher shows the group "her" system. When everyone seems clear, Jean asks, "Does anyone remember where this problem came from?" A girl raises her hand and says: "That was my problem a long time ago." "You're right," her teacher responds. "You asked that problem during the first week of school when all of you were asked to each bring a ream of paper for the year. You saw all those reams of paper stacked up in front of the room, and you wanted to know how many sheets of paper we had. I told you that we would find out some day but that at that point in the year it was hard to figure it out because you had to learn a lot about grouping, and adding large numbers. But now you all can do it and in many different ways." Another child recaps by noting, "That means that we used 14,000 sheets of paper this year," Jean says, "You got it!" The problem stays on the board for the day, along with the students' multiple solutions.

This first-year teacher's practice demonstrates that she understands how to organise a developmentally supportive classroom so that young children are productively engaged in meaningful work.

From Horowitz, Darling-Hammond, and Bransford (2005)

Mathematics teaching for diverse learners creates a space for the individual and the collective

Establishing norms of participation

Quality teaching facilitates students' growing awareness of themselves as legitimate creators of mathematical knowledge. Yackel and Cobb (1996) make the important observation from their research that the daily practices and rituals of the classroom play an important part in how students perceive and learn mathematics. Students create 'insider' knowledge of mathematical behaviour and discourse from the norms associated with those daily practices. This knowledge evolves as students take part in the "socially developed and patterned ways" (Scribner & Cole, 1981, p. 236) of the classroom. By scaffolding the development of those patterned ways, the teacher regulates the mathematical opportunities available in the classroom. Cobb, Wood, and Yackel (1993) have found in their research that cognitive development begins with a taken-as-shared sense of the expectations and obligations of mathematical participation.

How and when does the teacher set up practices that will contribute to mathematical thinking? Wood (2002) researched six classes over a two-year period, investigating the patterns of interaction within the classrooms. From data collected on a daily basis during the first four weeks of school, Wood examined the ways in which the six teachers set up the social norms for classroom interaction. Further data were gathered at a later date to compare and contrast discursive interactions when the same instructional activity took place in different classrooms. Wood found variation in students' ways of seeing and reasoning and these were attributed in the first place to the particular differences established in classrooms early in the year concerning *when* and *how to contribute* to mathematical discussions and *what to do as a listener*. The discourse principles propping up classroom participation regulated the selection, organisation, sequencing, pacing, and criteria of communication. Varying classroom expectations and obligations served to create marked differences in the cognitive levels demanded of the students and in the opportunities the students were given to engage in justification, abstraction, and generalisation. In the following vignette, Wood reveals how one teacher's early expectations of how seven-year-olds would participate in class discussion became a reality within her everyday teaching.

The 52 – 33 Problem

- Teacher: What did you get for an answer for this problem?
- Fred: 25.
- Sara: 19.
- Adam: 21.
- Teacher: Any other answers? Okay. Fred tell us how you got 25.
- Fred: We used the unifix cubes. 52 and then we took away 33. First I took away the tens. 42, 32, 22. Then I counted back the ones, 21, 20, 19.
- Karen: But you said it was 25.
- Fred: I know, but now I think it is 19, because I counted it again with the cubes.
- Teacher: John, what do you want to say? (He has his hand raised).
- John: I went back to 52 take away 30 is 22 (points to second problem on the paper). And I took away 3 more and that was 19. So I think it is 19.
- Teacher: Okay. But why did you take away 3?
- John: Because 52 take away 30 is 22, and 33 is 3 more than 30 so it was 19.
- Teacher: How did you know that it was 19?
- John: Because if 52 take 30 is 22, 52 take away 33 is 3 more than 30, so then I had to take away 3 more from 22, and that would be 19.
- Teacher: That makes sense. Sarah, what would you like to say?
- Sara: Well if you take 30 from 50, then you would have 20. Then you would have 2 and that 3, so you could take 1 from 20, and that would be 19.
- Mark: This is too confusing for me. Sarah, I don't understand why you took the 1 from 20.
- Sara: Because you have 2 minus 3 and so you need 1 off the tens.
- Mary: But if you took 1 from the 20, what happened to the 2 and the 3?
- Sara: I took 1 from the tens and added it to the 2 to make 3. [Then] 3 minus 3 is 0. So then I had to take 1 from the tens—20, and that makes the answer 19.
- Ryan: Well if you check it by adding 19 and 33, you get 52, so 19 is the answer.
- Karen: I think the answer must be 19, because we did it so many different ways to figure it out. And we got 19.
- Class: Agree. It is 19.

From Wood (2002)

Establishing participation processes and responsibilities for class discussion is an important pedagogical strategy. Classroom expectations and obligations concerning who might speak, when and in what form, and what listeners might do was the focus of a study by Nathan and Knuth (2003). One teacher's classroom was studied over a two-year period. During the second year, the teacher worked at creating social norms surrounding behaviour and participation in mathematical discussion. More student-centred than in the first year of the research, the established discourse principles facilitated students' participation in classroom interactions and ensured that students shared their thinking and listened attentively to each other. For example, the teacher told the class: "If you don't have an opinion, will you try and get one so we can keep this [discussion] going a little longer" (p. 195). As a result of the changed norms of participation, there was a marked increase in student contribution. However, the researchers point out that the teacher's focus was merely on changing the social norms of participation. Although the teacher sustained the interactions, it was with a view to keeping the conversation going rather than to nudge the conversation in mathematically enriching ways.

Students involved in the New Zealand Numeracy Development Project (NDP) have revealed that they like to be actively involved and they like to share their mathematical thinking with others (Young-Loveridge, Taylor, & Hawera, 2005). Students who were not involved in the NDP were less inclined to articulate the merits of listening to other students' strategies. NDP students, on the whole, saw an advantage in solving challenging problems, explaining personal solutions to their peers, as well as listening to and trying to make sense of someone else's explanations. Honouring students' contributions is an important inclusive strategy. Yackel and Cobb (1996) found that classroom teachers who facilitate student participation and elicit student contributions, and who invite students to listen to one another, respect one another and themselves, accept different viewpoints, and engage in an exchange of thinking and perspectives, are teachers who exemplify the hallmarks of sound pedagogical practice. In their exploration into the teaching practice of one teacher, McClain and Cobb (2001) reported that although the teacher accepted all students' ideas, it was an acceptance that did not differentiate between the mathematical integrity of those ideas. A pedagogical practice that does not attempt to synthesise students' individual contributions (Mercer, 1995) does not advance mathematical thinking. These findings support the earlier theoretical work of Doyle (e.g., Doyle & Carter, 1984) on classroom participation, which found that teachers use the strategy of accepting all answers as a way of achieving student cooperation in an activity.

Some students, more than others, appear to thrive in whole-class discussions. In their respective research, Baxter, Woodward, and Olson (2001) and Ball (1993) have found that highly articulate students tend to dominate classroom discussions and tend to offer valuable insights to the mathematical conversation. Typically, low academic achievers remain passive; when they do participate visibly, their contributions are comparatively weaker and their ideas sometimes muddled. Quality teaching ensures that participation in classroom discussion is safe for all students that the norms of student participation and contribution are equitable.

From their research, Planas and Gorgorió (2004) illustrate the ways in which teachers sometimes unwittingly create inconsistent social norms of participation. The study was undertaken in Spain, investigating social interactions at the beginning of the school year in a secondary mathematics classroom with a high percentage of immigrant students. Unlike local students, immigrant students were not permitted to participate in mathematical argumentation and hence did not have personal experience of how participation could help clarify and modify thinking. The researchers observed the teacher's 'subtle, systematic refusal' of immigrants' attempts to explain and justify their strategies for solving problems. Instead of participating actively, the immigrant students were required to remain passive observers while local students explained their thinking. As Planas and Gorgorió report, the reduced social obligations and lesser cognitive demands placed on these students had the effect of excluding them from full engagement in mathematics and hence constrained their development of a mathematical disposition.

Differential access to knowledge and the production of a mathematical identity was also reported by Clark (1997) and by Boaler, William, and Brown (2000). From their longitudinal study involving six UK schools, Boaler and colleagues document that lower streamed classes followed a protracted curriculum and experienced less varied teaching strategies. This curriculum polarisation had a marked effect on the students' sense of their own mathematical identity. The findings of these researchers are supported by other international research into the detrimental effects of such policies on the teaching and learning of students in lower streams (e.g., Gamoran, 1992; Slavin, 1990).

Working in groups

Many researchers have shown that small-group work can provide the context for social and cognitive engagement. Slavin (1995), in his meta-analysis of research on group learning, found a median effect size of .32 across studies using small groups of four heterogeneous members. These positive effects were characteristic not only of mathematics but also of other curriculum subjects. In another meta-analysis of the effects of small group learning in mathematics, Springer, Stanne, and Donovan (1997) found that such processes have significant and positive effects on undergraduates. The effect sizes recorded (.51), in fact, exceeded the average effect of .40 for classroom-based interventions on student achievement, as noted by Hattie, Marsh, Neill, and Richards (1997).

Thornton, Langrall, and Jones (1997) illustrate from a small study how classrooms organised for group work can provide a rich forum for diverse students to develop their mathematical thinking. They cite a study by Borasi, Kort, Leonard, and Stone (1993) in which a student who had a severe motor disability in writing, in addition to a 'numerical' disability, learned from his peers about how to share ideas and articulate his thinking. The student offered his explanation about a tessellating problem to the researcher and completed the recording task with support and questioning from the researcher over two days. As the researchers note:

This one-on-one work seemed really important and productive ... It was our hope that this experience would show [the student] what he could really do, and provide a model for the future; we do not expect him now to be able to do similar writing on his own yet, but perhaps he might be able to do it a second time around with less help, and gradually learn to do the same without the adult support. (p. 152)

Research has shown that gifted students, as well as low attainers, benefit from collaboration with peers. From a study involving six mathematically gifted students, Diezmann and Watters (2001) provide evidence that small homogeneous group collaboration significantly enhanced knowledge construction. Group participation also developed students' sense of self-efficacy. In particular, collaborative work that was focused on solving challenging tasks produced a higher level of cognitive engagement than that produced by independent work. The supportive group provided a forum for the giving and taking of critical feedback and building upon others' strategies and solutions. From their investigation, Diezmann and Watters claim that the positive effects of homogenous groupings for gifted students outweigh those offered through heterogeneous arrangements.

Doyle (1983) provides a theoretical grounding and empirical evidence for the ways in which effective groups operate. Successful group process depends on (a) the spatial configuration and interdependencies among participants, (b) how familiar the students are with the activity, (c) the rules established and the teacher's managing skills, and (d) students' inclinations to participate and their competencies. It is the teacher's responsibility to ensure that roles for participants, such as listening, writing, answering, questioning, and critically assessing, are understood and implemented. Cohen (1986) maintains that "if the group is held accountable for its work, there will be strong group forces that will prevent members from drifting off task" (p. 17). Cohen's extensive research into small-group effectiveness reveals that groups should be small in size. In particular, groups of four or five tend to be most effective. They should be

mixed in relation to academic achievement and any status characteristic. They need space for easy interactions and freedom from distractions.

The New Zealand NDP provides a teaching model that encourages both whole-class and small-group teaching. Through interaction with others within these teaching groups, the intention is that students will gradually develop the skills of and dispositions towards mathematically accepted ways of thinking and reasoning. Higgins (2005) has documented the way in which a teacher involved in the NDP used group organisation effectively as an instructional tool to enhance student engagement. The teacher set up the learning environment as a collective of groups working at varying levels of mathematical sophistication, each contributing to the overall class discussion and debate.

Groups provide opportunities to work with and learn from peers

Advocates of grouping claim that the organisational practice gives every student the opportunity to articulate thinking and understanding without every classroom eye and mind on what is being said. Wood and Yackel (1990) provide examples of how, in the course of working through problems with others, students extended their own framework for thinking. Benefits accrued as they listened to what their peers were saying and tried to make sense of it and coordinate it with their own thoughts on the situation. Whitenack, Knipping, and Kim (2001) also recorded the advantages for students when they explored the teacher's role in sustaining and enabling classroom mathematical practices in a second grade classroom. They noted how the teacher recognised important aspects of the students' interpretations as they worked individually, or with partners or in whole-class discussions. A collective conceptual shift was made by the class as they began to talk in ways that were similar to the quality explanations contributed by peers.

White (2003) found that students with limited English were more inclined to share their thinking with a friend rather than with the whole class. The teacher noted: "A lot of time they won't share something with the whole group. But they will share it with somebody sitting next to them, or they can sometimes get ideas from other kids who are sitting next to them" (p. 42). Peers serve as an important resource for developing mathematical thinking and for finding out about the nature of task demands and how those demands could be met (Doyle, 1983). Quality teaching pays attention to the important fact that students' willingness to contribute in the public arena of the classroom is influenced not only by the nature of the community established; it is also "affected by a student's ability to function in social situations and interpret the flow of events in a discussion. For some students the social skills needed for classroom lessons are not necessarily fostered at home or other nonschool settings" (Doyle, p. 180).

Artzt and Yaloz-Femia (1999) have shown that collaborative activity within a small supportive environment allows students not only to exchange ideas but also to test those ideas critically. Through groups, they learn to make conjectures as well as learn to engage in argumentation and validation. Helme and Clarke (2001) found in their secondary school classroom study that peer interactions, rather than teacher-student interactions, provide opportunity for students to engage in high-level cognitive activity. These researchers stress the important role the teacher plays in establishing social rules governing participation. They point out the way in which those rules serve to regulate the way cognitive engagement is taken up and expressed.

Baxter, Woodward, Voorhies, and Wong (2002) explored student group processes in one classroom over a seven-month intervention period. Of the 28 students in the class, two were categorised as at-risk and one other received the assistance of a teacher aide. The intervention was focused on the academic development of these low achievers. The target students participated in different mixed-ability groupings during small-group discussions. The teacher aide's key responsibility during these discussions was to provide support for the target students to actively participate in group discussions. Specifically, she ensured that they understood the problem and, where necessary, adapted the level of difficulty for them. She made sure that

they listened to the contributions of others, that they offered their own contributions, and that they could articulate the group's strategy for solving the problem. Baxter et al. report that the students were exposed to a wide range of ideas, strategies and solution pathways from their more academically able peers. Their peers' more advanced cognitive levels provided richer social-emotional as well as cognitive outcomes for the target students than would have been possible in a remedial classroom setting.

This same finding is reported in Alton-Lee (2003). In the *Quality Teaching for Diverse Students in Schooling: Best Evidence Synthesis*, Alton-Lee reports that the teachers in a study by Webb (1991) who set aside time to instruct students about the intricacies of effective group processes invariably enhanced students' outcomes. Students who learned to help each other learned that to make the group work effective, communication and feedback within the group needed to be centred on mathematical explanations and justifications rather than on single answers to problems. Alton-Lee concludes that "Webb's finding is particularly important because it shows the potential benefit to high achievers as well as low achievers" (p. 28).

Holton and Thomas (2001) have called two-way peer tutoring "reciprocal scaffolding" (p. 99). They note that students interacting need to have sufficient competence and experience to allow them to ask appropriate questions of themselves and each other. Rawlins (in progress) finds that although students feel comfortable asking the teacher for help, their first preference is to discuss mathematical problems with their peers. Year 12 students studying for the National Certificate of Educational Achievement (NCEA) reported in a group interview:

- S1: We discuss how we arrived at our conclusion. For example, in algebra we talk about how we interpreted the application question and how we turned that information into a number problem.
- S2: Normally if we have a question wrong we compare and look at the others working to find out where we went wrong. Then we discuss how we view the questions.
- S3: Just what their answer was and how they got it. Normally we try to help each other out as much as we can.

Similarly, in a study of year 12 students learning calculus, Walshaw (2005) reports the same value placed by students in peer tutoring. As one student says:

Kate and I study a lot together and we help each other because she understands. She seems to understand a lot more of it than I do. I don't get much of it at all this year. It's just going straight past me. But she's understanding it so she helps me out with that. We help each other and if she doesn't understand it then I'd ring one of the guys who are friends. Martin would help. He got like 94% in School C. So I'd ring him. My older brother would help but he's not hugely good with maths. I mean, he just scraped through last year, so he sort of helps me with what he can but he gets to a point where it's beyond him and he can't cope with it so I have to turn to friends who I know will be able to help me with it. And it's fresh in their minds. (p. 26)

Goos (2004) documents the way in which one student views his interactions with peers in his senior mathematics class as an enriching learning experience:

Adam helps me ... see things in different ways. Because, like, if you have two people who think differently and you both work on the same problem you both see different areas of it, and so it helps a lot more. More than having twice the brain, it's like having *ten* times the brain, having two people working on a problem (p. 278).

The following vignette shows Adam guided the learning of Luke and helped him see his error.

Adam and Luke Supporting Each Other's Learning

The students were investigating the iterative processes underlying fractals and chaos theory. The class had considered the example of the Middle Thirds Cantor Set, a fractal constructed by starting with a line of length 1, removing the middle third, then removing the middle third of the remaining segments and repeating this process infinitely many times. The point of the example was to prove that the sum of all lengths removed is equal to the length of the original line, a surprising and counterintuitive result. Students were then asked in a subsequent activity to find how much space is removed from the Middle Fifths Cantor Set. A common error made by many students was simply to substitute 1/5 for 1/3 in the proof provided in the worked example. The worked example for Middle Thirds Cantor Set was available to the students. Adam and Luke are investigating the problem.

- Luke: It's going to be a fifth instead of a third [pointing to example, no response from Adam].
- Adam: It's going to be a fifth [points to example].
- Adam: Just think ... start, work through it from the beginning.
- Luke: I am. [Writes.]
- Adam: [Glances at Luke's work.] Work through it! [Emphatically]
- Luke: [Not looking up.] I am. [Looks up, puzzled.] What am I doing? [Checks example.] The size remaining's right, isn't it? [Adam looks at Luke's work and chuckles.] That's right!
- Adam: OK, you just do it.
- Luke: OK, fine, whatever.
- Adam: [Opens his own book and checks his working, grins at Luke.] Wrong!
- Luke: [Expression of disbelief on his face, looks at his working.] How and where? I cannot see where it could possibly be wrong!
- Adam: [Pauses, raises eyebrows, makes the decision to rescue Luke.] OK, explain this to me. Explain ... explain this to me [pointing to Luke's working].

From Goos (2004)

By way of contrast, Thomas (1994) explored the primary school classroom to find out what primary students actually say when they are engaged in classroom talk. In analysing the talk of 46 New Zealand primary school students during mathematics lessons, Thomas reported that "in the many hours of recorded and transcribed talk there were few instances of the children engaged in talk which could be directly linked with learning in the sense of a child obviously understanding something as a result of their talk with another child" (p.ii).

Peter-Koop (2002) provides evidence of students' refusal to interact with others. In a study that explored group processes in third and fourth grade classrooms, the researcher demonstrated the difficulty that students can have in engaging with a new line of thought, given the distractions of group discussion. A New Zealand study undertaken by Higgins (1997) revealed that young students' group work (new entrant to J2) was not as effective as their teachers believed it to be. Higgins showed that student explanations appeared to be constrained by the group process. In later research, Higgins (2000) demonstrated that teachers are often unclear about their role during student group work. However, when the mathematical intent of the group activity was articulated at the beginning and again during the feedback episode, and when student contributions were evaluated in terms of that intent, students appeared to engage more actively with the mathematics.

In her investigation into classroom group processes at the senior secondary school level, Barnes (2005) found that both poor social relationships and poor communication within groups contributed to limited student mathematical engagement in an activity. Barnes analysed video data to explore precisely who introduced ideas, the response of others, and and who controlled,

sustained, or impeded the discussion. She provides evidence that two students were frequently interrupted and their efforts ignored by others during group work. These ‘outsiders’ were assigned their position by others who did not recognise their rights to explain, question, or challenge. Barnes reports that these students learned less, and although the video data revealed that they offered the group distinctive mathematical insight, these students tended to lose confidence in their mathematical ability. Barnes suggests that pedagogical practice that regularly includes all students in group work reinforces the norms of careful and courteous listening.

Individual thinking time

A number of studies have provided evidence of the benefits for some students of independent learning approaches. For example, McMahon (2000) reported on successful individualised teaching in a Mathematics Recovery Programme targeting year 2 students. The two teachers involved had undertaken a year-long programme to develop purposeful instructional resources and strategies for their individualised teaching sessions. They focused the level appropriately, encouraged independent learning, and gave the students time to reflect on their thinking and methods. The success of the intervention depended crucially on the teachers’ sophisticated craftsmanship, which involved both anticipating and supporting students’ responses. The researchers report that the students increased their confidence in their ability and their mathematical understanding.

Walshaw (2004) reports on one student who advanced her learning more through independent thinking than through collaborative efforts with peers. A belief in the value of student collaboration to enhance mathematical learning was at the forefront of research undertaken by Sfard and Kieran (2001). The researchers report on two students, Gur and Ari, who were set a task by the teacher and expected to work together towards producing a solution. Classroom observations led the researchers to believe that the students were working together but further scrutiny revealed otherwise:

While having a close look at a pair of students working together, we realised that the merits of learning-by-talking cannot be taken for granted. Our analyses compel us to conclude that if Gur did make any real progress, it was not *thanks* to his collaboration with Ari but rather *in spite* of it, and if this collaboration did, in the end, spur Gur’s development, it was probably mainly in an indirect way, by providing him with an incentive to learn. Our experiment has shown that the interaction between the two boys was unhelpful to either of them. The present study, therefore, does not lend support to the common belief that working together can always be trusted to have a synergetic quality. It is not necessarily true that two people who join forces can do more than the sum of what each one of them can do alone. (p. 70)

Sfard and Kieran showed how articulating mathematical thinking to oneself can have beneficial effects for the individual. These highly respected researchers conceptualise ‘talking to oneself’ as a form of communication and record from their research how an invisible and inaudible discourse with self creates mathematical thinking. They make the point that “interaction with others, with its numerous demands on one’s attention, can often be counterproductive. Indeed, it is very difficult to keep a well-focused conversation going when also trying to solve problems and be creative about them” (p. 70).

Students need some time alone to think and work quietly away from the demands of a group. Reliance on classroom grouping by ability may have a detrimental effect on the development of a mathematical disposition and students’ sense of mathematical identity. What effective teachers do is create a space for both the individual and the collective. They use a range of organisational processes to enhance students’ thinking and to engage them more fully in the creation of mathematical knowledge. More significantly, over and above establishing structures for participation, the effective teacher constantly monitors, reflects upon, and makes necessary

changes to those arrangements on the basis of their inclusiveness and effectiveness for the classroom community.

Mathematics teaching for diverse learners demands explicit instruction

Shaping students' mathematical language

Quality teaching bridges students' intuitive understandings and the mathematical understandings sanctioned by the world at large. Language plays a central role in building these bridges: it constructs meaning for students as they move towards modes of thinking and reasoning characterised by precision, brevity, and logical coherence (Marton & Tsui, 2004). The teacher who makes a difference for diverse learners is focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics. In order to enculturate their students into the mathematics community, effective teachers share with their students the conventions and meanings associated with mathematical discourse, representation, and forms of argument (Cobb & Yackel, 1996; Wood, 2002). Khisty and Chval (2002), among others, have reported that the language that students use derives from the language used by their teacher. The responsibility for distinguishing between terms and phrases and sensitising students to their particular nuances weighs heavily with the teacher, who profoundly influences the mathematical meanings made by the students.

Competency in mathematics allows the student to understand discussions about mathematics, to learn the subject, and to comprehend the mathematical way of speaking. It demonstrates control over the specialised discourse (Gee & Clinton, 2000). But the specialised language of mathematics can be problematic for learners. Particular words, grammar, and vocabulary used in school mathematics can hinder access to the meaning sought and the objective for a given lesson. Words, phrases, and terms can take on completely different meanings from those that they have in the everyday context. Walkerdine (1988), for example, has reported the difficulties young students encounter in establishing the mathematical meaning of 'less than' and 'more', given their idiosyncratic meanings within specific home practices. Similarly, Christensen (2004) has provided evidence that, in some cases, mathematical words can assume quite different meanings in everyday life. He documents the difficulties encountered by Te Poutama Tau students in using *mua* and *muri* for number sequencing, and in using *whakamua* and *whakamuri* for forward and backward directional counting.

Developing this point for Australian students, Sullivan, Mousley, and Zevenbergen (2003) found that students with a familiarity of standard English (usually children from middle-class homes) had greater access to school mathematics. As the teachers in their study said, the students were able to 'crack the code' of the language being spoken. One teacher of students from non-English-speaking backgrounds makes the point about meanings of words: "[Y]ou need to reinforce: 'Tell me what I mean when I say estimating?' or 'Where are some things that you estimate?' Ground it in their world because for a child for whom English is not their first language, if there are numbers they'll be right, but if you say 'estimating' they won't have a clue what that might mean" (p. 118).

Initiating and eliciting

A fruitful approach, aimed at clarifying descriptions and explanations, is for the teacher to purposefully provide information or ask questions. Lobato, Clarke, and Ellis (2005) refer to this as an aspect of teacher 'telling' (p. 102). The form of 'telling' that Lobato, Clarke, and Ellis advocate is one that facilitates learning by initiating student reflection on the concept and on the process. The approach is directed at developing students' conceptual knowledge rather than their memory skills. This form of telling does not take away from students the agency for making sense of mathematics (Hiebert & Wearne, 1993). Lobato, Clarke, and Ellis (2005)

develop the pedagogical approach further. They expand on teacher telling by describing two strategies. The first, they call *initiating*, by which they mean a group of actions and behaviours used to introduce new ideas that function as prompts for the way in which students construct mathematics. The purpose is conceptual: it is not aimed at showing procedural steps, rather, its intent is to shape ideas and make connections between ideas in a coherent and sensible fashion. The manner in which this is done includes the use of symbols, images and graphics, the summarising of student work and adding in of new material, counter-examples, questioning, and the providing of new representations. Lobato, Clarke, and Ellis differentiate this approach from a second strategy, which they call *eliciting*. Following on from initiation, the teacher *elicits* information to determine how the students have interpreted the mathematical concept.

Initiating and eliciting pedagogical strategies are illustrated in studies by Turner and colleagues (1998, 2002). These researchers found that what distinguished high-involvement year 5 and 6 classrooms was the engagement of the teachers in forms of instruction focused on shaping students' understandings. In particular, teachers negotiated meaning through 'telling' tailored to students' current understandings. They shared and then transferred responsibility so that students could attain greater autonomy. They also fostered intrinsic motivation by sparking curiosity and by supporting students' goals. In these classrooms, telling was followed by a pedagogical action that had the express intent of finding out students' understandings and interpretations of the given information.

Multilingual contexts

A number of studies have investigated the challenges of teaching mathematics in multilingual contexts. Adler (2001), Khisty (1995) and Moschkovich (1999), for example, have all studied the tensions that arise in multilingual classrooms between mathematics and language, and have explored the teacher's role in this relationship. Neville-Barton and Barton (2005) looked at these tensions as experienced by Chinese Mandarin-speaking students in New Zealand schools. Their investigation focused on the difficulties that could be attributable to limited proficiency with the English language. It also sought to identify language features that might create difficulties for students. Two tests were administered, seven weeks apart. In each, one half of the students sat the English version and the other half sat the Mandarin version, ensuring that each student experienced both versions. There was a noticeable difference in their performances on the two versions. On average, the students were disadvantaged in the English test by 15%. What created problems for them was the syntax of mathematical discourse. In particular, prepositions, word order, and interpretation of difficulties arising out of the contexts. Vocabulary did not appear to disadvantage the students to the same extent. Importantly, Neville-Barton and Barton found that the teachers of the students in their study had not been aware of some of the student misunderstandings.

Like the students in the study undertaken by Neville-Barton and Barton (2005), Pasifika students in [Latu's \(2005\)](#) research had difficulty with syntax. Word problems involving mathematical implication and logical structures such as conditionals and negation were a particular issue for students from senior mathematics classes. They also found technical vocabulary, rather than general vocabulary, to be problematic. Latu noted that English words are sometimes phonetically translated into Pasifika languages to express mathematical ideas when no suitable vocabulary is available in the home language. The same point was made by Fasi (1999) in his study with Tongan students. Concepts such as 'absolute value', 'standard deviation', and 'simultaneous equations' and comparative terms like 'very likely', 'probable', and 'almost certain' have no equivalent in Tongan culture, while some English words, such as 'sikuea' (square), have multiple Tongan equivalents. The suggestion is that special courses in English mathematical discourse be delivered with the express intent of connecting the underlying meaning of a concept in English with the students' home language.

Other pedagogical strategies for multilingual classrooms have been advocated by Cohen (1984). Cohen has found that small-group work can assist in overcoming potential and real

language difficulties. Students with limited English proficiency need to hear their peers use mathematical language. The most effective way of doing this is to place students in groups with English speakers and a proficient bilingual student. To ensure the effectiveness of the group and the enhancement of mathematical knowledge of all its members, the teacher needs to clarify the important role that the bilingual student will play in bridging understandings between members. Cohen reports on research that found that in classrooms in which two languages were used by the teacher and students, bilingual students enjoyed the highest social status.

Home language and code switching

Latu (2005) records that both Sāmoan and Tongan students encountered problems with relational statements. Fasi investigated the discursive approaches of two teachers, one Sāmoan and the other Tongan, both of whom had been educated in their native country before moving to New Zealand to complete their higher education. He found that the teachers switched between the language of instruction and the learners' main language in order to explain and clarify the concepts to students. Adler (1998), Setati and Adler (2001), Dawe (1983), and Clarkson (1992) all found evidence of language switching (code switching) for bilingual students, particularly when students could not understand the mathematical concept or when the task level increased. Code switching involved words and phrases as well as sentences and tended to enhance student understanding.

In Latu's study, low general proficiency in both home language and in the language of instruction was predictive of students' difficulty with mathematics. This finding is supported by Christensen (2004), who reported a positive correlation between pāngarau achievement and language proficiency. Clarkson and Dawe (1997) also found that, compared with monolingual students, bilingual students with low proficiency in both the home language and the language of instruction are at a cognitive disadvantage and achieve less successfully. However, De Avila (1980) offered an alternative conclusion. In a study of year 1, 3, and 5 Hispanic students, De Avila found that language proficiency was not strongly related to mathematics achievement. Linguistic proficiency is unlikely by itself to contribute to poor performance. Other factors, such as 'linguistic distance' (Dawe, 1983) between the home and instructional languages might help to explain the relationship.

Researching in South Africa, Adler (2001) has highlighted a number of dilemmas experienced by teachers in that country. One is whether to focus explicitly on correct (accepted-at-large) mathematical language or to focus on mathematical reasoning. Teachers have found that a focus on language form compromises the mathematical content and vice versa. Woodward and Irwin (2005) explored this issue when they observed two teachers of year 5 and 6 students, mostly Pasifika, involved in the New Zealand Numeracy Development Project. Under investigation was the teachers' use of vocabulary in probability lessons and the linguistic structures employed in discursive interactions between teachers and students and between peers. The researchers report that the language that dominated the lessons was everyday English and, as a consequence, little effort was expended on teaching students the nuances of mathematical language. It would have been particularly useful for these students if the teachers had been able to capture the subtleties of mathematical language.

Within our theorising of a productive learning community, mathematical language involves more than vocabulary and technical usage. In this section, we have seen that mathematical language encompasses the ways that expert and novice mathematicians use language to explain and to justify a concept. Listening carefully to students' expressions of mathematical content is a sound first-step pedagogical strategy. Quality teaching puts the spotlight on mathematics; this focus directly implicates students' relationships, both with the concepts and with each other.

Mathematics teaching for diverse learners involves respectful exchange of ideas

Students' articulating thinking

There is now a large body of empirical and theoretical evidence that demonstrates the beneficial effects of students articulating their mathematical thinking (e.g., Fraivillig, Murphy, & Fuson, 1999; Lampert, 1990; O'Connor, 1998). By expressing their ideas, students provide their teachers with information about what they know and what they need to learn. Fraivillig and colleagues have found that effective teachers do more than sustain discussion; they nudge conversations in mathematically enriching ways, they clarify mathematical conventions and they arbitrate between competing conjectures. In short, they pick up on the critical moments in discursive interactions and take learning forward. Hiebert and colleagues (1997) have found that relevant and meaningful teacher talk involves drawing out the specific mathematical ideas encased within students' methods, sharing other methods, and advancing students' understanding of appropriate mathematical conventions. Reframing student talk in mathematically acceptable language provides teachers with the opportunity to enhance connections between language and conceptual understanding.

Articulating comprehensible explanations about mathematical concepts is a learned strategy. The effective teacher is aware that "the art of communicating has to be taught" (Sfard & Kieran, 2001, p. 70). It is, however, a major challenge to make classroom discourse an integral part of an overall strategy of teaching and learning (Hicks, 1998; Lampert & Blunk, 1998). Doing so successfully involves significantly more than developing a respectful, trusting and non-threatening climate for discussion and problem solving. Quality teaching involves socialising students into a larger mathematical world that honours standards of reasoning and rules of practice (Popkewitz, 1988). O'Connor and Michaels (1996) put it this way:

The teacher must give each child an opportunity to work through the problem under discussion while simultaneously encouraging each of them to listen to and attend to the solution paths of others, building on each others' thinking. Yet she must also actively take a role in making certain that the class gets to the necessary goal: perhaps a particular solution or a certain formulation that will lead to the next step ... Finally, she must find a way to tie together the different approaches to a solution, taking everyone with her. At another level just as important she must get them to see themselves and each other as legitimate contributors to the problem at hand. (p. 65)

Quality teaching familiarises students with the rules that regulate appropriate mathematical ways, including inference, analysis, and modelling. O'Connor and Michaels (1996) highlight the importance of shaping students' higher-level thinking by fostering students' involvement in taking and defending a particular position against the claims of other students. This instructional process depends upon the skilful orchestration of classroom discussion by the teacher. It is a skill that effective teachers have developed. It "provides a site for aligning students with each other and with the content of the academic work while simultaneously socialising them into particular ways of speaking and thinking" (p. 65).

Stigler (1988) looked at teachers who have developed the skill of scaffolding the responses of diverse students. He compared the pedagogical approaches of Japanese and American teachers and found that Japanese teachers spend more time than American teachers in encouraging their students to produce comprehensive verbal explanations of mathematical concepts and algorithms. Expanding on this aspect, Cobb, Wood, and Yackel (1993) report that effective teachers in their research initiated and guided a genuine mathematical dialogue between students. These teachers made it possible for students to share their interpretations of tasks and their solutions. In addition, the teachers influenced the course of the dialogue by picking up on students' contributions. They did this by framing students' interpretations and solutions as topics for discussion. Valuing and shaping students' mathematical contributions served these important functions:

- it allowed students to see mathematics as created by communities of people;
- it supported students' learning by involving them in the creation and validation of ideas;
- it helped students to become aware of more conceptually advanced forms of mathematical activity.

White (2003) explored how two teachers used classroom discourse to teach third graders mathematics, looking at how the discourse enhanced the educational experiences of the teachers' diverse student populations and how it influenced the mathematical thinking of the students. The teachers were part of larger project, IMPACT (Increasing the Mathematical Power of All Children and Teachers). Classroom vignettes illustrated one of four themes that emerged from the classroom discourse: (a) valuing students' ideas, (b) exploring students' answers, (c) incorporating students' background knowledge, and (d) encouraging student-to-student communication. The teachers engaged all students in discourse by first monitoring their participation in discussions and then deciding when and how to encourage each to participate. By actively listening to students' ideas and suggestions, they demonstrated the value they placed on each student's contribution to the thinking of the class. The teachers encouraged their students to give critical feedback on each other's responses and asked them to reveal their assessment of each other's ideas by giving a 'thumbs up' or 'thumbs down' signal. In one of the classrooms for limited-English-speaking students, this proved to be a particularly useful strategy for getting members to share their views with the class.

Hunter (2005) examined the discourse patterns within a low-decile year 5 classroom of predominantly Pasifika and Māori students. In this vignette, a range of pedagogical strategies is utilised by the teacher to support a productive mathematical discourse:

Clarifying Expectations of Classroom Discourse

The teacher's specific pedagogical effectiveness within classroom discussion was her use of explicit strategies to enhance the mathematical contributions of her students:

- If you don't understand, what questions do you need to ask?
- If someone didn't understand it though and the same thing was said to them ...
- I want you to explain to the people in your group how you think you are going to go about working it out. Then I want you to ask if they understand what you are on about and let them ask you questions. Remember in the end you all need to be able to explain how your group did it so think of questions you might be asked and try them out.
- Okay so I have heard lots of talking, discussing in your groups and listening to each other and that's good.
- Now this group is going to explain and you are going to look at what they do and how they came up with the rule for their pattern, right? Then as they go along if you are not sure please ask them questions. Tune in here, step by step, and as they go along if you can't make sense of each step remember ask those questions.
- Pen down. Have a look and think. Now has anyone got a question they want to ask of Rewa at this point?
- Arguing is not a bad word ... sometimes I know you people think to argue is ... I am talking about arguing in a good way. So please feel free if you do not agree with what someone has said as long as you say it in an okay way. A suggestion could be that you might say I don't actually agree with you, could you show that to me. Do you think you could prove it mathematically, could you, perhaps write it, or draw something to show that idea to me ... and sometimes doing that the other person thinks it wasn't quite right so they change their idea and that's okay.

From Hunter (2005)

Hunter's research reveals a pedagogical strategy that presses students for understanding. It is a strategy that aims to develop thinking and speaking in ways that are mathematical. Students are encouraged and supported in developing the skills of explanation, argumentation, and justification. Other New Zealand studies have found that students are not always able to elaborate on their mathematical reasoning. Meaney (2005) explored the responses of 35 students to questions in the 2003 Mathematics with Statistics national examination paper that required mathematical justification. The ways in which students constructed their mathematical explanations varied. Twelve students used both equations and narrative explanation to explain their thinking, while 17 used only equations. In an investigation into primary school students' responses to tasks requiring explanations of commutativity, Anthony and Walshaw (2002) noted that many students were unable to offer explanations that reflected structural understanding. Moreover, several students experienced unease with the expectation that they justify their thinking. At the secondary level, Bicknell (1998) studied students' written explanations and justifications for mathematical assessment tasks. She writes of the 36 year 11 mathematics students involved in her study:

[S]tudents experienced some difficulties writing explanations and had concerns about whether their explanations were satisfactory; ... most students surveyed were unable to write justifications; they lacked knowledge and confidence in justifying their solutions. (p. ii)

Lubienski (2002), as teacher-researcher, compared the learning experiences of students of diverse socio-economic status (SES) in a seventh grade classroom. She reported that higher SES students believed that the patterns of interaction and discourse established within the classroom helped them learn other ways of thinking about ideas. The discussions helped them reflect, clarify, and modify their own thinking, and construct convincing arguments. In Lubienski's view, the lower SES students were reluctant to contribute because they lacked confidence in their ability. They claimed that the wide range of ideas contributed in the discussions confused their efforts to produce correct answers. Their difficulty in distinguishing between mathematically appropriate solutions and nonsensical solutions influenced their decisions to give up trying. Pedagogy, in Lubienski's analysis, tended to privilege the ways of being and doing of high SES students. In a similar way, Jones's (1991) classic study showed that the discursive skills and systems knowledge that are characteristic of high SES families align them favourably with the pedagogy that is operationalised within school settings. Set in the New Zealand context, Jones provided conclusive evidence that Pasifika girls were unwittingly penalised by the sorts of instructional approaches taken by classroom teachers.

Opportunities for students to explain and justify solutions

The benefits of providing regular opportunities for students to explain and justify their solutions are well documented. Many researchers have found that pedagogical practices that make provision for the development and evaluation of mathematical argument and proof contribute to the development of students' mathematical thinking. At another level, such provision provides access to an individual's mathematical thinking. When an individual's thinking is accessed by a learning community, the intentions and interpretations of the individual lend themselves to modification in response to the community's reception of their ideas (White, 2003).

In the New Zealand context, **Woodward and Irwin (2005)** report on opportunities for students to explain and justify solutions. A particular teacher involved in their study made a significant contribution to students' mathematical development. She did this by listening attentively to her students' queries and explanations, asking them to justify their answers, and holding back with explanations until she deemed them crucial. The researchers record one of many occasions during which she structured students' mathematical practice: "Before you write it down I want you to justify it to your partner. So if you say there's eight queens your partner needs to say, 'How do you know that there's eight queens?'" (p. 803). By using such pedagogical strategies, the teacher was able to develop mathematical ways of doing and being in all her students.

McChesney (2005) explored levels of student participation in low- and middle-band New Zealand classes at the junior secondary school level. McChesney notes that teachers who established classroom communities in which there was access to social, discursive, visual, and technological resources, were able to support students' mathematical activity. Her research clearly demonstrates a direct relationship between the quality of teacher–student interaction and students' negotiation of mathematical meaning. The effective teachers in this research were able to set up an environment in which conventional mathematical language migrated from the teacher to the students. Over time, students' contributions, which were initially marked by informal understandings, began to appropriate the language and the understandings of the wider mathematical community. It was through the take-up of conventional language that mathematical ideas were seeded.

Within the intellectual space that is shared by students and their teacher, it is the teacher's pedagogical content knowledge and expertise that makes a difference to the quality and level of mathematical discussion. What needs to be stressed, too, is that the teacher's expertise is also related to the role he or she assumes as earnest listener and co-learner. Zack and Graves (2002) have reported that teachers who make a difference are themselves active searchers and enquirers into mathematics. O'Connor's (2001) classroom research highlights how one teacher, through purposeful listening, facilitated a group of students towards a mathematical solution. The discussion in the following vignette illustrates how students took positions on the answer and attempted to support those positions with evidence. The teacher made her contribution by challenging the students' claims, using counter-examples.

Gina and Bruno's Alternative Conceptions

A class has been discussing whether fractions can be converted into decimals. One student (Bruno) directly addresses the logic of the framing question: to successfully argue that not all fractions can be turned into decimals, you need an example of a fraction that cannot be turned into a decimal. Almost immediately, a claim he made is challenged. Gina offers the counter-example of $\frac{2}{5}$.

- Gina: I disagree because five can go, uh, any fraction with five or fifths in it, can go into, can also be turned into a decimal.
- Teacher: Give me an example.
- Gina: Um, two fifths is, is one tenth, umm, two fifths is umm is turned into ...
- Teacher: Okay, what does two fifths look like as a decimal?
- Gina: Point forty.

The teacher asks Bruno to respond, and he immediately replies that fifths are covered under his original statement: fraction denominators that are factors of powers of ten (like fifths) all do allow transformation of fractions to decimals. (So his claim could have been stated more precisely, e.g.: 'any odd prime number that is not also a factor of a power of ten will result in a repeating decimal.')

At this point the class looks a bit stunned at the directness of Gina's counter-example, and the alacrity with which her challenge is returned. There is silence. Gracefully, the teacher takes this opportunity for a meta-comment that depicts the two as collaborating in an important mathematical practice:

- Teacher: Great, now I hope you're listening because what Gina and Bruno said was very important. Bruno made a conjecture and Gina tested it for him. And based on her tests he revised his conjecture because that's what a conjecture is. It means that you think that you're seeing a pattern so you're gonna come up with a statement that you think is true, but you're not convinced yet. But based on her further evidence, Bruno revised his conjecture. Then he might go back to revise it again, back to what he originally said or to something totally new. But they're doing something important. They're looking for patterns and they're trying to come up with generalisations.

The teacher has sensed that this excellent display of the mathematical practice of finding and testing counter-examples to a claim is not without its potential risks. By portraying the exchange as positive, she is reassuring all students that the practice does not connote hostility, as it might in any informal or everyday setting.

From O'Connor (2001)

Goos (2004) described how a secondary school mathematics teacher developed his students' mathematical thinking through scaffolding the processes of enquiry. There was a tacit agreement amongst members of the learning community that instructional practices demanded students' mathematical talk. For his part, the teacher orchestrated mathematical events by first securing student attention and participation in the classroom discussion. Specifically the "teacher call[ed] on students to clarify, elaborate, critique, and justify their assertions. The teacher structured students' thinking by leading them through strategic steps or linking ideas to previously or concurrently developed knowledge" (p. 269). In a series of lesson episodes, Goos provides evidence of how the teacher pulled learners "forward into mature participation in communities of mathematical practice" (p. 283). He scaffolded thinking by providing a predictable structure for enquiry through which he enacted his expectations regarding sense making, ownership, self-monitoring and justification. As the year progressed, the teacher gradually withdrew his support to push students towards more independent engagement with mathematical ideas. For their part, the students responded by completing tasks with decreasing teacher assistance and by proposing and evaluating alternative solutions. Engagement for them was "a complex process that combine[d] doing, talking, thinking, feeling, and belonging" (Wenger, 1998, p. 56).

Constructive feedback

Constructive feedback, as one form of exchange of ideas, has a powerful influence on student achievement (Hattie, 2002). In keeping with our ethic of care, we have evidence that praise, not of itself but taken together with quality feedback, can be a powerful pedagogical strategy (Hill & Hawk, 2000). Of course, in an environment that does not value student contribution or knowledge, feedback has decidedly negative effects (Hoyles, 1982).

What constitutes quality feedback? Research has shown that feedback that engages learners in further purposeful knowledge construction will contribute to the development of their mathematical identities. We have evidence (e.g., Wiliam, 1999) that feedback that is constructive has the effect of occasioning certain mathematical capabilities in students and assists in the development of their perception of the mathematical world.

Khisty and Chval (2002) have shown that quality feedback plays a key role in students' learning. In their research into the way in which a teacher interacted with her fifth grade Latino students (and with English-language learners), the researchers found that her focus on mathematical talk and meaning enabled the students to develop mathematical reasoning in significant ways. She facilitated learning through questioning that was concerned less about teacher exposition and more about the perceptions held by her students. The teacher opened up the discussion with each interaction and, by making use of the responses received, she was able to lend structure to their mathematical meanings.

Teachers who provide quality feedback draw on a range of pedagogical content knowledge skills that enable them to know when to and when not to intervene. When they do intervene, their feedback makes a judgment about students' strategies, skills, or attainment (Gipps, McCallum, & Hargreaves, 2000), pinpointing the difference "between the actual level and the reference level of a system parameter which is used to alter the gap in some way" (Ramaprasad, 1983, p. 4). In other words, the feedback makes a comparison between where a student is currently at and a standard as interpreted by the teacher. We can classify as feedback much of the dialogue that occurs in the classroom to support learning (Askew & Lodge, 2000). Feedback is a rich

resource by which students are able to gauge and “monitor the strengths and weaknesses of their performances, so that aspects associated with success or high quality can be recognised and reinforced, and unsatisfactory aspects modified or improved” (Sadler, 1989, p. 120).

This is not to advocate grades and test scores for enriching student learning, or to promote feedback as the responsibility of the teacher alone. In her study involving senior secondary school students, Anthony (1996) found that, by itself, feedback in the shape of scores and grades was of limited value. She makes a case for pedagogical practices that shift total responsibility from the teacher and include student participation in the feedback process. She recommends that teachers be explicit about the value of student reflection and that students be given the opportunity to evaluate the reasonableness of their solutions and assess the justification for their procedures.

Feedback that is provided too early or too late can be ineffective. Freeman and Lewis (1998) have demonstrated that students require an opportunity to confront the mathematical problem themselves before they are given feedback and that “the greater the delay, the less likely it is that the students will find it useful or be inclined to act on it” (p. 49). Knight (2003) interviewed six New Zealand teachers and observed the feedback they provided in their primary school numeracy classes. The study revealed that out of the 349 examples of verbal feedback recorded in the lessons, most (83%) took the form of an expression of encouragement or praise. Knight reports that the teachers were often unaware of the high frequency of such responses and their automated nature. Only 17% of the feedback reflected on the cognitive development of the students. As Knight notes, a more effective pedagogical strategy would aim to support students’ mathematical thinking as well as their motivation.

Classroom research at both primary and secondary level (e.g., Ruthven, 2002; Wiliam, 1999) has shown that much of the teacher feedback that students receive is not particularly constructive. Ruthven reported that, in the UK, teacher feedback was not assisting students “to study mathematics and think mathematically” (p. 189). Rawlins (in progress) reports that year 12 students working towards the National Certificate of Educational Achievement (NCEA) valued teacher feedback that “pointed them in the right direction.” In the following transcript, two of the participant students discuss their views on what quality feedback involves:

- S1: Just stating where I went wrong ... “This is what you should have done ...”
- S2: Like when we went back through them and she said, “OK, next Monday I want you to hand them in done” but she gave us the answers for each of the questions. I found that hopeless. I couldn’t figure out how they got to the answer and I was just sitting there an ...
- S1: You sit there for about an hour on your calculator and you do all sorts of thing ...
- S2: ... and you just cannot figure it out and I found that really useless.
- S1: There were a couple of questions—the really crazy ones—where she did the first bit for us and then told us what to do next, like you need to factorise, expand, solve, and the ...
- S2 ... that was good. That was helpful. But she didn’t do that for all of them.

Wiliam (1999) reported on a comparative study of two groups of year 4 students. One group was given minimal feedback that nevertheless allowed members to advance towards a solution to a mathematics problem. The other group was given a full solution when they reached the point where they could not make further progress without help. Wiliam found that “[m]inimal intervention promoted better learning” (p. 9). As Chamberlain (2005) found in a study involving teachers of the middle grades, minimal feedback tends to more effectively promote student learning and retention. These teachers urged: “Allow the students to work on the problem

statement. As they work, your role should be one of a facilitator and observer. Avoid questions or comments that steer the students toward a particular solution” (p. 165).

Too much feedback is counterproductive to learning. This point is illustrated by **Woodward and Irwin (2005)**. Their study compares the talk and feedback provided by two teachers of mostly Pasifika year 5 and 6 students involved in the New Zealand Numeracy Development Project. For one of the teachers, the researchers recorded numerous instances of too much feedback or ‘teacher lust’ (Maddern & Court, 1989). While this teacher had created a positive learning environment and shown a keen desire for talk to occur in the classroom, actual mathematical talk was minimal. Cognitive space was limited by the lack of pause times for thinking, and students were occasionally ‘talked over’. Specifically, students did not have the opportunity to learn and speak the language of mathematicians. Like the teacher in a study by Khisty and Chval (2002), this teacher did not provide students with opportunities to engage in mathematical discourse so they did not develop a medium for expressing what they were learning. Khisty and Chval found that although the teacher in their study had done many of the ‘proper’ things, by contextualising the mathematics in a story that had relevance to the students, by developing a role-playing activity to assist with conceptual understanding, and by forming small working groups, they nevertheless failed to provide students with the meanings associated with mathematical knowledge.

In reporting on their study of classroom dialogue, Cobb, Wood, and Yackel (1993) describe how one teacher shaped student participation, giving it the cultural nuances of mathematical talk:

[The teacher] reformulated their explanations and justifications in terms that were more compatible with the mathematical practices of society at large and yet were accepted by the children as descriptions of what they had actually done. Thus rather than funnelling the children’s contributions, the teacher took the lead from their contributions and encouraged them to build on each others’ explanations as she guided conversations about mathematics. As a consequence, the mathematical meanings and practices institutionalised in the classroom were not immutably decided in advance by the teacher but, instead, emerged during the course of conversations characterised by ... a genuine commitment to communicate. (p. 93)

Revoicing

‘Revoicing’ is the term used by O’Connor and Michaels (1996) to describe a subtle yet effective strategy for fine-tuning mathematical thinking. By revoicing is meant the repeating, rephrasing or expansion of student talk in order to clarify or highlight content, extend reasoning, include new ideas, or move discussion in another direction. The researchers maintain that probing into student understanding provides teachers with the opportunity to model engagement within a mathematical, multi-voiced community. According to Forman and Ansell (2001), in classrooms where revoicing is used, “[t]here is a greater tendency for students to provide the explanations ... and for the teacher to repeat, expand, recast, or translate student explanations for the speaker and the rest of the class” (p. 119). In the following vignette, revoicing is used by a teacher of year 1 and 2 students as a simple yet effective pedagogical strategy for scaffolding knowledge.

Five Monkeys in the Tree

Students in a year 1-2 classroom had been puzzling over the problem: “If all the monkeys in a big tree and a small tree want to play in the trees, think of all the ways that we can see all five monkeys in the two trees.”

Together the students had provided the following possibilities: 5,0; 2,3; 3,2; 0,5; 4,1; 1,4. The teacher responded:

Teacher Is there a way that we could be sure and know that we’ve gotten all the ways?

- Jordan: [Goes to the overhead screen and points to the two trees and the table as he explains.] See, if you had four in this [big] tree and one in this [small] tree in here, and one in this [big] tree and four in this [small] tree, couldn't be that no more. If you had five in this [big] tree and none in this [small] tree, you could do one more. But you've already got it right here [points to 5,0]. And if you get two in this [small] tree and three in that [big] tree, but you can't do that because three in this [small] one and two in that [big] one—there is no more ways, I guess.
- Teacher: What Jordan said is that you can look at the numbers and there are only a certain ... there are only certain ways you can make five.

From Cobb, Boufi, McClain, and Whitenack (1997)

Providing cognitive structure and fine-tuning mathematical thinking

O'Connor and Michaels (1996) provide evidence that teachers who provide cognitive structure also tend to fine-tune students' mathematical thinking. Fraivillig, Murphy, and Fuson (1999) have developed a conceptual framework for describing the ways by which teachers do this. The three key pedagogical components identified in their Advancing Children's Thinking (ACT) framework are 'eliciting', 'supporting', and 'extending'. Eliciting involves promoting and managing classroom interactions, supporting involves assisting individuals' thinking, and extending captures those practices that work to advance students' knowledge.

Shaping students' mathematical thinking is a highly complex activity (Taylor & Cox, 1997). It is complex because teachers and students are "negotiating more than conceptual differences ... they are building an understanding of what it means to think and speak mathematically" (Meyer & Turner, 2002, p. 19). Building that understanding requires the teacher to first construct sociomathematical norms (see Yackel & Cobb, 1996) for what constitutes a mathematically acceptable, different, sophisticated, efficient, or elegant explanation. Sociomathematical norms regulate mathematical argumentation and govern the learning opportunities and ownership of knowledge made available within the classroom.

Manouchehri and Enderson (1999) investigated the discursive interactions within a heterogeneously grouped seventh grade mathematics class. cursory observations revealed an overwhelming occurrence of student talk and interaction. Further analysis unpacked the teacher's critical role in orchestrating that mathematical activity and discourse. Through careful questioning, purposeful interventions, and her efforts to shift the students' reliance from her towards "the guidance, support and challenge of companions who vary in skills and status" (Rogoff et al., 1993, p. 5), she provided responsive rather than directive support, all the while monitoring student engagement and understanding. Her strategy was not aimed at structuring learning by organising students' behaviour. Rather, as Manouchehri and Enderson (p. 219) clarify, her primary objectives were to:

- facilitate the establishment of situations in which students had to share ideas and elaborate on their thinking (e.g., Would anyone else like to add anything to S13's explanation? Could you show that to us on the board? That is an excellent question. Does anyone want to have a shot at it?);
- help students expand the boundary of their exploration (e.g., What do you think class? Do you think that this formula would work all the time for all the rows? Why don't you extend the sequence and see if there is a pattern?);
- encourage students to make connections among different discoveries and develop a deeper understanding of the interrelationships among the patterns that students identified (e.g., I wonder if we can find out how these 2 patterns are related?);
- invite multiple representations of ideas (e.g., Is there another way of representing this?).

The way in which one teacher orchestrated a year 1 and 2 classroom discussion is illustrated by Fraillig, Murphy, and Fuson (1999) in the following vignette. What is particularly effective is the way the teacher sustains the discussions. She has developed a sensitivity about when to ‘step in and out’ (Lampert & Blunk, 1999) of the classroom interactions and has learned how to resolve competing student claims and address misunderstanding or confusion (theirs and hers). For their part, the students listen to others’ ideas and debate to establish common meanings. In short, they participate in a ‘microcosm of mathematical practice’ (Schoenfeld, 1992), learning how to appropriate mathematical ideas, language and methods and how to become apprentice mathematicians.

High-level Thinking of Young Mathematicians

Ms Smith challenged her year 1–2 class to investigate zero under subtraction. Her students’ mathematical development was influenced by several factors—the discursive interactions, the multiple forms of mathematical representation, and particularly by the way she explicitly nudged her students towards and pressed for understanding and demonstrations of mathematical behaviour. The teacher drew on her mathematical content knowledge to make specific links between concepts. By asking them questions, listening attentively, encouraging them to look for relationships amongst concepts, checking for accuracy, and building on students’ ideas to stimulate further thought, she nurtured a mathematical disposition that incorporated risk-taking and multiple solutions. Talk in this young learners’ classroom was not merely aimed at filling a conversational void nor directed at easy solution pathways, but was strategically focused towards conjecturing and high-level thinking.

- Teacher: What’s the last one? [Ms. Smith was eliciting all single-digit “doubles” from students and listing them on the board.]
- S1: Zero plus zero.
- Teacher: Zero plus zero.
- S2: Zero plus zero is just zero.
- Teacher: Zero plus zero is easy.
- S3: Zero minus zero is negative one, isn’t it?
- Teacher: No. Derrick [S3]’ You’re on zero [points to the number line] and you take zero jumps. Where are you [giving one context for deciding the answer]?
- S3: Zero. [Ms. Smith motions to S3 indicating “you got it.”]
- S4: Then negative five plus negative five must be negative five.
- Teacher: Pardon me?
- S4: Negative five plus negative five should be negative five.
- Teacher: No, ‘cuz you’re adding negative five and negative five, so you start at negative five and how many jumps do you take?
- S4: Five.
- Teacher: Well, you’re not going to end up on five [points to the negative five on the number line]; you’re not going to end up on negative five [modifying her sentence]. So, then negative five. How many jumps do you take?
- S4: Five.
- Teacher: So where are you going to end up?
- S: Zero plus ...
- Teacher: No, no, no. Negative five [pause indicating uncertainty]. You’re right, Stevie [S4]. You’re right (laughs). You see what he did? Ms. Smith was thinking the other way. Negative five (pause); ... Allan, this is hard. You might want to watch it for a minute. Negative five [she continues slowly with a questioning voice], we’re going to add negative five to it. No, it’s not right. Is it?

- S5: [Students speculate as to whether or not zero is the answer.] Yeah, zero's right.
- S6: No, it's ten. Negative goes that way [motioning toward negative ten, meaning negative five plus negative five equals negative ten].
- Teacher: No, it's ten [responding to S5]. But you're adding negative five to it, sweetie, so you would go this way [motions toward the left of the number line]. I was right. It's ten. You start at negative five plus five, you end up at zero. But negative five plus negative five is negative ten. I was right. [Ms. Smith whispers seemingly to reassure herself.] I was right.
- S: You thought you were wrong!
- Teacher: I did think I was wrong. You confused ...
- S: Negative five and negative five is negative ten.
- Teacher: That's right. Negative five plus negative five is negative ten.

From Fraivillig, Murphy, and Fuson (1999)

Pedagogical practices like these, which help refine students' mathematical thinking, are reported in research undertaken by Steinberg, Empson, and Carpenter (2004). In their study, a respectful exchange of ideas was central to a sustained change in students' conceptual understanding. Over a period of a few months, the teacher had integrated particular pedagogical strategies into her practice, focused on probing and interpreting student understanding and on generating new knowledge. During the study, she developed a working consensus with all members of the classroom community about the form of, and social roles within, her changed instructional processes. A respectful exchange of ideas was also a feature of effective pedagogical practice in a study by Forman and Ansell (2001). The teacher in this study used repetition to highlight the particular claims and ideas of individual students, developed the understandings implicit within those ideas, negotiated meaning to establish the veridicality of a claim, and used their original ideas as a springboard for developing related new knowledge in whole-class discussions. Original ideas may be exchanged between an individual and the teacher. A New Zealand study (Walshaw, 2000) involving secondary school students documents how an exchange of ideas between teacher and student was effectively used to monitor the understanding and written work of individual students and to guide the teacher's choice of examples and explanations at the whiteboard.

Mathematics teaching for diverse learners demands teacher content knowledge, knowledge of mathematics pedagogy, and reflecting-in-action

Teacher knowledge

Effective teaching for diverse students begins with teacher knowledge. What teachers do in classrooms is very much dependent on what they know and believe about mathematics. It is also very dependent on what they understand about the teaching and learning of mathematics (Hiebert et al., 1997). Jaworski (2004) argues that, to be successful, a teacher must have both the *intention* and the *effect* to assist students to make sense of mathematical topics: good intentions are necessary, but they are not enough. The teacher must make good sense of the mathematics involved or he or she will not be able to help students work with ideas and knowledge (Fraivillig et al., 1999; Shifter, 2001). In this section, we explore the effects of teacher knowledge on student outcomes and we develop these ideas further in chapter 5.

Expanding on the crucial importance of teachers' subject knowledge as a resource for teaching, Ball and Bass (2000) point out that there is an intimate relationship between mathematical concepts and the way those concepts are actually conveyed by the teacher to students. The

findings of studies undertaken by Ball and Bass (2000) and many others (e.g., Carpenter & Lehrer, 1999; Doerr & Lesh, 2002; Hill, Rowan, & Ball, 2005; Kilpatrick et al., 2001; Ma, 1999; Shulman & Shulman, 2004; Warfield, 2001) signal that teachers must have sound content knowledge if they are to access the conceptual understandings that students are articulating in their methods and if they are to decide how those understandings might have come about and where they might be heading.

A core infrastructural element of effective mathematics pedagogy is *knowledge* in the subject area (Fraser & Spiller, 2001). In their study, Bliss, Askew, and Macrae (1996) found that when teachers demonstrated limited or confused understanding of the subject knowledge relevant to the lesson, or when their perception of the rationale or execution of the curriculum was not clear, their students struggled to make sense of the mathematical concepts. Teachers who were unclear in their own minds about the mathematical ideas struggled to teach those ideas and often used examples and metaphors that prevented, rather than helped, student development. Where teachers were insecure in their content knowledge, there was a direct, subsequent student lack of understanding. The teachers in the study acknowledged that their own limited knowledge led them to misunderstand their students' solutions and to give feedback that was inappropriate or unhelpful.

In a study of whole-class teaching episodes at three schools, Myhill and Warren (2005) found that many strategies used by teachers worked more as devices to enable students to complete tasks rather than as learning support mechanisms that would help move them towards independence. In the year 2 and year 6 lessons observed, teachers often used 'heavy prompts', pointing students to the 'right' answers. Doyle and Carter (1984) note that this pedagogical strategy is sometimes called 'piloting' and is often used as an inclusive strategy to ensure that every student has the opportunity to provide a correct answer. However, using this strategy, the teachers in the Myhill and Warren study tended to miss critical opportunities for gaining insight into students' prior knowledge or level of understanding. Teachers' fragile subject knowledge prevented them from assessing the current level of mathematical sophistication and put boundaries around the ways in which they could develop students' responses. One year 6 teacher told her class: "Sometimes in decimals you say point seven eight or sometimes you say point seventy-eight", and in doing so, paved the way for student misunderstanding of place value.

Knowing how to teach the content

As well as documenting the importance for student outcomes of strong subject matter (content) knowledge, research has demonstrated the importance of mathematics pedagogical knowledge, that is, teachers' *knowledge of how to teach the content*. It is one thing to access the level of conceptual sophistication that students are working with, and it is another to know how to utilise that content knowledge (Hattie, 2002; Hill et al., 2005). On the basis of findings from their research, Ball and Bass (2000) emphasise the important work that the teacher does in connecting mathematical ideas in real time with instructional approaches, teaching principles, and students' contributions. Quality teaching integrates knowledge flexibly in varying contexts. As Ball and Bass and others (e.g., Hill et al., 2005; Schifter, 2001; Warfield, 2001) have made clear, teachers' decisions about activities always fall back on the knowledge that they hold of the content to be taught.

Teachers making connections

Askew, Brown, Rhodes, Johnson, and Wiliam (1997) looked closely at the repertoire of mathematical knowledge that characterised effective teachers of numeracy in the UK. Their Effective Teachers of Numeracy Project revealed that quality teaching is not necessarily related to higher formal qualifications. In fact, the researchers found a slight negative relationship between the level of teachers' formal qualifications in mathematics and the levels of attainment of their students. Evidence from case studies signalled that teachers who were highly qualified

in mathematics tended to have a more procedural view of school mathematics. In contrast, it was the teachers who were able to make connections between aspects of mathematical knowledge who recorded high academic gains for their students. The Effective Teachers of Numeracy Project revealed that “highly effective teachers of numeracy themselves had knowledge and awareness of conceptual connections between the areas which they taught in the primary mathematics curriculum” (Askew et al., 1997, p. 3). Teachers’ conceptual understanding and knowledge is critically important at any level. In New Zealand secondary schools, the 2001 Teacher Census found the proportion of teachers with a third-year university or postgraduate qualification in a given subject area to be higher for mathematics than for any other curriculum area.

Thomas, Tagg, and Ward (2002) reported on numeracy teaching in New Zealand. In a questionnaire undertaken by 148 teachers involved in the NDP, almost all (96%) maintained that their knowledge had developed through their involvement in the project. In turn, the new knowledge led to effectiveness in their teaching, as evidenced through student gains that exceeded prediction in the five aspects of number learning assessed. These achievement gains were recorded irrespective of students’ gender, age, ethnicity, or school decile rating. Specifically, teachers noted that as a result of their involvement in the NDP, changes to their practice included:

- a greater focus on both strategies and knowledge using the structure of the Number Framework;
- more effective assessment strategies and student grouping decisions;
- an increased focus on listening to students and their explanations;
- more sharing of student strategies;
- a greater understanding of student progression.

Thomas and Ward (2002) carried out case studies of 10 teachers who were identified as effective with respect to student achievement in early number. Despite their varying levels of teaching experience, all had sound knowledge of their students’ mathematical knowledge and learning capabilities, positive and enriching relationships with their students, and high expectations for their students and were thorough in their understanding and application of the Number Framework. Amongst other things, the teachers said:

He will put up his hand as he is really keen, and then he will forget what he is saying, or he won’t know, so I will usually wait longer, or I say “Keep thinking and I will come back to you.” Or I get to him one-to-one and find out what he is doing that way.

Asking students to explain their thinking and waiting for them to do so. Sometimes I get a bit impatient and want to butt in. So you are trying. Sometimes you are trying to cover too much, it’s best to just cover a little bit so that *you do have time to do that*.

... to go from bundling them [popsticks] up to “OK, show me this number,” that is a step in itself ... a small step, but it is still a step ... That was what I was trying to do ... go from me modelling it, to them doing it, and then take it a step further ... from a given set of equipment to a number, “Now show me this number.” (pp. 4–49)

Davies and Walker (2005) focused on the content knowledge that teachers bring to their teaching. In an experiment aimed at enhancing teacher knowledge and improving student numeracy levels, these researchers explored how teacher knowledge and pedagogical practices changed as a result of collaboration between teachers and researchers. At each meeting, the eight teachers introduced, discussed, and trialled rich tasks and problems that challenged their ‘content knowledge complexes’. As a result of their more finely tuned classroom listening and questioning skills, teachers began to notice changes in their teaching behaviours. In addition, the teachers volunteered that their planning had changed and that they had developed an increased sensitivity to the need to wait for students’ responses.

Askew and Millett (in press) investigated the relationship between teacher knowledge and student thinking. They looked particularly at critical incidents of teaching/learning—those moments that occasioned students’ thinking. What became obvious during the course of the research was the critical role that teachers’ subject knowledge plays in extending and challenging students’ conceptual ideas. Sound subject knowledge enabled teachers to mediate between the mathematical tasks, the artefacts, the talk, and the actions surrounding the teaching/learning critical incidents. Askew and Millett found that teachers with limited subject knowledge tended to focus the class on a narrow conceptual field rather than on forging wider connections between the facts, concepts, structures, and practices of mathematics.

On-the-spot reflection

As Askew and Millett observed, pedagogical practice that makes a difference for all learners requires professional reflecting-in-action. It requires a moment-by-moment synthesis of actions, thinking, theories, and principles (Ball & Bass, 2000). In the research undertaken by Askew and Millett, the teachers who were able to develop student mathematical understanding were those that had a sound base of subject knowledge. This knowledge informed their on-the-spot decision making in the classroom. It informed decisions about the particular content that the students would learn, the activities they carried out, how they engaged with the content, and how they conveyed to the teacher their understanding of the content.

In her research with teachers, Sherin (2002) found that they negotiate between three areas of knowledge: their understanding of subject matter, their perception of curriculum materials, and their personal theories of student learning. As they weave between these three areas of knowledge and as they deepen their own understanding of them, effective teachers are able to increase students’ levels of mathematical knowledge. The negotiation that takes place as teachers reflect in action, draws on a rich history of personally established ways of thinking and being and applying knowledge flexibly (Hattie, 2002). In particular, teachers who reflect in action are able to adapt and modify their routine practices and, in the process, contribute to the development of new pedagogical routines and new knowledge about subject matter.

Quality teaching involves a positive teacher stance towards reflecting-in-action (Schifter & Fosnot, 1993). It needs to be pointed out that reflecting-in-action can take place either within (Sherin, 2002) or beyond (Davies & Walker, 2005; Mewborn, 1999; Wood, 2001) the classroom. Jaworski (1994) offers first-hand accounts of reflecting-in-action within mathematics classrooms. Contrary to scholarly critique that claims such practice is impossible on the grounds that the metacognitive activity involved assumes more time than the classroom could possibly offer, Jaworski (2004) provides evidence of teachers noticing and then acting knowledgeably as they interact at critical moments in the classroom when students create a moment of choice or opportunity.

Teachers who reflect in action and negotiate between forms of knowledge work hard at understanding students’ viewpoints. Teachers do not, however, always accurately anticipate student thinking and behaviour. Nathan and Koedinger (2000) documented the ways in which teachers predicted students’ problem-solving activity. The researchers found that “students’ problem-solving behaviours differ in a systematic way from those predicted by teachers” (p.184). Teachers believed that story problems and word-equation problems would be more difficult for their students than symbol-equation problems. The research students found that symbolically presented problems posed more difficulty. Bennett (2002) provides evidence that this kind of problem also creates difficulties for New Zealand secondary school students.

An effective teacher tries to delve into the minds of students by noticing and listening carefully to what students have to say (Franke & Kazemi, 2001). Jones (1986) describes how one teacher in a secondary school class encouraged a student to think through a problem:

Student: I can’t see why that $2x$ should be there.

Teacher: Go back to the last line. Can you explain what was done there?

Student: Yes. I can see how you had to divide by x squared to get there.

Teacher: So what did you need to do to go from there? (p. 484)

Yackel, Cobb, and Wood (1999) report on the ways in which one year 2 teacher listened to, reflected upon, and learned from her students' mathematical reasoning while they were involved in a discussion on relationships between numbers. Analyses of the discussion revealed that her mathematical subject knowledge and her focus on listening, observing, and questioning for understanding and clarification greatly enhanced her understanding of students' thinking.

Listening and Noticing: $37 + 23 + 14$

The class is discussing the problem: $37 + 23 + 14$. Three children have already given explanations as to how they arrived at 74 for a sum. In each case, the solution involved decomposing the summands into tens and ones and recombining them in various ways. The episode begins just after Latoya reported that she got 73 as an answer and the teacher asked her to explain her solution.

Latoya: I said the 10 on the 14, and I added, and then I added 30 more, that was 40.

Teacher: Where did you get the 30 from?

Latoya: Out of the 23.

The teacher assumes that Latoya added the 10 from the 14 and the 20 from the 23 to get 30 so he asks a clarifying question.

Teacher: Okay, you added, you added this 10 and this 20?

Latoya: Yeah ... and I got 40.

Teacher: You got 40.

Latoya: No, I said that I had 10 and I added 30 more.

Tonya: Where did you get the 30 from?

Latoya: Out of the 23 ... And $30 + 30$ equal to 60.

Several students enter the discussion to ask Latoya about her apparent interpretation of the 3 in 23 as 30.

Carmen: How did you get the 30 out of the 23?

Latoya: Um, I said that I took away the 20 and left the 3 there, and added another 3.

Lemar: Well, how come did you take away that 2? When you said, take away that 2 and that left 3 and 3 and that was 30?

Latoya: Um.

Teacher: Maybe, let me put 23 down here, 23 (writes 23 on the chalkboard). They want to know how did you get 30 out of 23? That's what they're trying to figure out.

Latoya: Um.

Teacher: Listen, let me ask you a question. That 2 stands for what?

Latoya: 20.

Teacher: All right, and the 3 stands for 30?

Latoya: Yes.

Tonya: Why are you saying 20, you are saying that's a 20 and that's a 30? Why are you saying ... the 20 and the 30?

Teacher: Give her a chance to answer, please. She asked a nice question. She said, she said that's a, the 2 is a 20, and 3 was a 30, and she asked you if that's 20 - 30?

Latoya: No. The 2 stands for 20 and 3 stands for (long pause).

At this point in the episode, the teacher called on another student to "help her out."

Rick: That 2 stands for 2 tens, that makes 20, and that 3 is, that makes 3 ones.

Following this, the teacher assisted Latoya as she attempted to explain the problem. Together, they came just short of completing the sum before another student joined in to complete the explanation.

From Yackel, Cobb, and Wood (1999)

Conclusion

This chapter has explored how mathematics teachers create the conditions for learning in their classroom communities. We found that teaching that facilitates learning for diverse learners demands an ethic of care. Research in this area has found that effective teachers demonstrate their caring by establishing classroom spaces that are hospitable as well as academically 'charged'. They work at developing interrelationships that create spaces for students to develop their mathematical and cultural identities. Teachers who care work hard to find out what helps and what hinders students' learning. They have high yet realistic expectations about enhancing students' capacity to think, reason, communicate, reflect upon, and critique their own practice, and they provide students with opportunities to ask why the class is doing certain things and with what effect.

At the same time, research quite clearly demonstrates that pedagogy that is focused solely on the development of a trusting and attentive microculture does not get to the heart of what mathematics teaching truly entails. A context that supports the growth of students' mathematical identities and competencies creates a space for both the individual and the collective. Many researchers have shown that small-group work can provide the context for social and cognitive engagement. Quality teaching uses both individual and group processes to enhance students' cognitive thinking and to engage them more fully in the creation of mathematical knowledge. Within the classroom, all students need time alone to think and work quietly, away from the demands of a group. This line of research has also revealed that classroom grouping by ability has a detrimental effect on the development of a mathematical disposition and on students' sense of their own mathematical identity.

A teacher with the intention of developing student understanding will not necessarily produce the desired effect. Unless teachers make good sense of the mathematical ideas, they will not develop the flexibility they need for spotting the golden opportunities and wise points of entry that they can use for moving students towards more sophisticated and mathematically grounded understandings. Reflecting on the spot and dealing with contested and contesting mathematical thinking demands sound teacher knowledge. Studies exploring the impact of content and pedagogical knowledge have shown that what teachers do in classrooms and the way in which they manage multiple viewpoints is very much dependent on what they know and believe about mathematics and on what they understand about the teaching and learning of mathematics. A successful teacher of mathematics will have both the *intention* and the *effect* to assist pupils to make sense of mathematical topics. There is now a wealth of evidence available that shows how teachers' knowledge can be developed with the support and encouragement of a professional community of learners.

Studies have provided conclusive evidence that teaching that is effective is able to bridge students' intuitive understandings and the mathematical understandings sanctioned by the world at large. Consistently emphasised in research is the fact that teaching is a process involving analysis, critical thinking, and problem solving. Language, of course, also plays a central role. The teacher who makes a difference for diverse learners is focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics. Responsibility for distinguishing between terms and phrases and sensitising the students to their particular nuances weighs heavily with the teacher, who profoundly influences the mathematical meanings made by the students in the class. Classroom work is made more enriching when discussion involves the respectful exchange of ideas, when teachers ensure

that this exchange is inclusive of all students, and when the ideas put forward are (or become) commensurate with mathematical conventions and curricular goals. The effective teacher is able to orchestrate discussion and argumentation and facilitate dialogue, not only for the development of mathematical competencies and identities but also to ensure important social outcomes.

References

- Adler, J. (1998). A language of teaching dilemmas: Unlocking the complex multilingual secondary mathematics. *For the Learning of Mathematics*, 18, 24-33.
- Adler, J. (2001). *Teaching mathematics in multilingual classrooms*. Dordrecht: Kluwer.
- Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington: Ministry of Education.
- Angier, C., & Povey, H. (1999). One teacher and a class of school students: Their perception of the culture of their mathematics classroom and its construction. *Educational Review*, 51(2), 147-160.
- Anthony, G. (1996). Classroom instructional factors affecting mathematics students' strategic learning behaviours. In P. Clarkson (Ed.), *Technology in mathematics education* (Proceedings of the 18th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 38-48). Melbourne: MERGA.
- Anthony, G., & Walshaw, M. (2002). Swaps and switches: Students' understandings of commutativity. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.), *Mathematics Education in the South Pacific*. (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 91-99). Sydney: MERGA.
- Artzt, A.F., & Yaloz-Femia, S. (1999). Mathematical reasoning during small-group problem solving. In L.V. Stiff & F.R. Curcio (Eds.), *Developing mathematical reasoning K-12: 1999 Yearbook* (pp. 115-126). Reston, VA: National Council of Teachers of Mathematics.
- Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy*. London: Kings College.
- Askew, M., & Millett, A. (in press). Critical incidents: When teacher subject knowledge matters. In M. Brown, M. Askew & A. Millett (Eds.), *Learning about number: Interactions and outcomes in primary classrooms*. Netherlands: Kluwer Academic Publishers.
- Askew, S., & Lodge, C. (2000). Gifts, ping-pong and loops: Linking feedback and learning, In S. Askew (Ed.), *Feedback for learning* (pp. 1-17). London: Routledge.
- Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93(4), 373-397.
- Ball, D. L., & Cohen, D.K. (1997). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco, CA: Jossey-Bass.
- Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.
- Ballard, K. (2003). The analysis of context: Some thoughts on teacher education, culture, colonisation, and inequality. In K. Nes, M. Stromstad, & T. Booth (Eds.), *The challenge of inclusion: Reforming teacher education*. London: RoutledgeFalmer.
- Bana, J., & Walshaw, M. (2001). A window into mathematics communities of practice in Australia and New Zealand. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australia, Vol. 1, pp. 96-103). Sydney: MERGA.
- Barnes, M. (2005). 'Outsiders' in a collaborative learning classroom. In M. Goos, C. Kanes, & R. Brown (Eds.), *Mathematics education and society* (Proceedings of the 4th International Mathematics Education and Society conference, pp. 58-68). Brisbane: Griffith University.
- Bartholomew, H. (2003). Ability grouping and the construction of different types of learner in mathematics classrooms. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 128-135). Sydney: MERGA.
- Barton, B., Paterson, J., Kensington-Miller, B., & Bartholomew, H. (2005). Dodging the dragon. Strategies for mathematics professional development in low socio-economic areas. In M. Goos, C. Kanes, & R. Brown (Eds.), *Mathematics education and society* (Proceedings of the 4th International Mathematics Education and Society Conference, pp. 78-87). Brisbane: Griffith University.
- Baxter, J., Woodward, J., & Olson, D. (2001). Effects of reform-based mathematics instruction on low achievers in five third grade classrooms. *Elementary School Journal*, 101(5), 529-548.

- Baxter, J., Woodward, J., Voorhies, J., & Wong, J. (2002). We talk about it, but do they get it? *Learning Disabilities Research and Practice*, 17(3), 173-185.
- Bennett, P. (2002). *Word problems in teaching and learning algebra*. Unpublished MEdStuds (Mathematics) thesis, Massey University, Palmerston North, New Zealand.
- Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *Journal of Mathematics Teacher Education*, 8, 171-191.
- Bicknell, B. (1998). *The writing of explanations and justifications in mathematics*. Unpublished master's thesis, Massey University, Palmerston North, New Zealand.
- Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht: Kluwer.
- Bishop, R., & Glynn, T. (1999). *Culture counts: Changing power relations in education*. Palmerston North: Dunmore Press.
- Bliss, J., Askew, M., & Macrae, S. (1996). Effective teaching and learning: Scaffolding revisited. *Oxford Review of Education*, 22(1), 37-61.
- Boaler, J., Wiliam, D., & Brown, M. (2000). Students' experiences of ability grouping: Disaffection, polarisation and the construction of failure? *British Educational Research Journal*, 26(5), 631-648.
- Brooks, J., & Brooks, M. (1993). *In search of understanding: The case for the constructivist classroom*. Alexandria: Association for Supervision and Curriculum Development.
- Carpenter, T.P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding, In E. Fennema, & T.A. Carpenter (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum Associates.
- Chamberlain, M.T. (2005). Teachers' discussion of students' thinking: Meeting the challenge of attending to students' thinking. *Journal of Mathematics Teacher Education*, 8, 141-170.
- Christensen, I. (2004). *An evaluation of Te Poutama Tau 2003: Exploring issues in mathematics education*. Wellington: Ministry of Education.
- Clark, M. (1997). Comfort in the mathematics classroom. *New Zealand Mathematics Magazine*, 34(2), 14-22.
- Clarkson, P. (1992). Language and mathematics: A comparison of bilingual and monolingual students of mathematics. *Educational Studies in Mathematics*, 23(4), 417-430.
- Clarkson, P., & Dawe, L. (1997). NESB migrant students studying mathematics: Vietnamese students in Melbourne and Sydney. In E. Pehkonen (Ed.), *Proceedings of the 21st conference of the International Group for the Psychology in Mathematics Education* (Vol. 2, pp.153-160). Lahti, Finland: PME.
- Cobb, P., & Hodge, L.L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning*, 4(2&3), 249-284.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3/4), 175-190.
- Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28(3), 258-277.
- Cobb, P., Perlwitz, M., & Underwood-Gregg, D. (1998). Individual construction, mathematical acculturation, and the classroom community. In M. Larochelle, N., Bednarz, & J. Garrison (Eds.), *Constructivism in education* (pp. 63-80). Cambridge: Cambridge University Press.
- Cohen, E. (1986). *Designing groupwork: Strategies for the heterogeneous classroom*. New York: Teachers College.
- Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & C. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development*. Oxford: Oxford University Press.
- Davies, N., & Walker, K. (2005). Learning to notice: One aspect of teachers' content knowledge in the numeracy classroom. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematic Education Research Group of Australasia, Vol.1, pp. 273-280). Sydney: MERGA.
- Dawe, L. (1983). Bilingualism and mathematical reasoning in English as a second language. *Educational Studies in Mathematics*, 14, 325-353.
- De Avila, E. (1980). *Relative language proficiency types: A comparison of prevalence, achievement level, and socioeconomic status*. Report submitted to the RAND Corporation.
- Diezmann, C.M., & Watters, J. J. (2001). The collaboration of mathematically gifted students on challenging tasks. *Journal for the Education of the Gifted*, 25(1), 7-31.
- Doerr, H., & Lesh, R. (2002). A modelling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A model and modelling perspective on mathematics teaching, learning and problem solving* (pp. 125-139). Mahwah, NJ: Lawrence Erlbaum Associates.
- Doyle, W. (1983). Academic work. *Review of Educational Research*, 53(2), 159-199.
- Doyle, W., & Carter, K. (1984). Academic tasks in classrooms. *Curriculum Inquiry*, 14(2), 129-149.
- Fasi, U.M.L. (1999). Bilingualism and learning mathematics in English as a second language in Tonga. Unpublished Thesis, University of Reading, England.

- FitzSimons, G., Seah, W.T, Bishop, A., & Clarkson, P. (2001). Beyond numeracy: Values in the mathematics classroom. In J. Bobis, B. Perry, M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 202-219). Sydney: MERGA.
- Forman, E., & Ansell, E. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics*, 46, 114-142.
- Fraivillig, J., Murphy, L., & Fuson, K. (1999). Advancing children's mathematical thinking in *Everyday Mathematics* classrooms. *Journal for Research in Mathematics Education*, 30(2), 148-170.
- Franke, M.L. & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterising generative growth. In T. Wood, B.S. Nelson, J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 47-74). Mahawah, NJ: Lawrence Erlbaum Associates.
- Fraser, D., & Spiller, D. (2001). Effective teachers. In C. McGee & D. Fraser (Eds.), *The professional practice of teaching* (2nd ed., pp. 67-84). Palmerston North: Dunmore Press.
- Freeman, R., & Lewis, R. (1998). *Planning and implementing assessment*. London: Kogan Page.
- Gamoran, A. (1992). The variable effects of high school tracking. *American Sociological Review*, 57(6), 812-828.
- Gee, J. P., & Clinton, K. (2000). An African American child's science talk: Co-construction of meaning from the perspectives of multiple discourses. In M. A. Gallego & S. Hollingsworth (Eds.), *What counts as literacy: Challenging the school standard* (pp. 118-138). New York: Teachers College Press.
- Gipps, C., McCallum, B., & Hargreaves, E. (2000). *What makes a good primary school teacher?* London: Routledge Falmer.
- Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, 35(4), 258-291.
- Hackenberg, A. (2005). A model of mathematical learning and caring relations. *For the Learning of Mathematics*, 25(1), 45-51.
- Hattie, J. (2002). What are the attributes of excellent teachers? In B. Webber (Ed.), *Teachers make a difference: What is the research evidence?* (Proceedings of the New Zealand Association of Research in Education Conference, pp. 3-26). Wellington: NZARE.
- Hattie, J.A., Marsh, H. W., Neill, J.T., & Richards, G.E. (1997). Adventure education and outward bound: Out-of-class experiences that make a lasting difference. *Review of Educational Research*, 67(1), 43-87.
- Helme, S., & Clarke, D. (2001). Identifying cognitive engagement in the mathematics classroom. *Mathematics Education Research Journal*, 13(2), 133-153.
- Hicks, D. (1998). Closing reflections on mathematical talk and mathematics teaching. In M. Lampert & M. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 241-252). New York: Cambridge University press.
- Hiebert, J., & Wearne, D. (1993). Instructional tasks: Classroom discourse and students' learning in second-grade arithmetic. *American Educational Research Journal*. 30(2), 393-425.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Higgins, J. (1997). Constraints on peer explanations in junior mathematics classrooms. *SAMEpapers: Science and Mathematics Education Papers*, 128-145.
- Higgins, J. (2000). "I don't know how much to interfere." Independent group work and teacher interaction in the junior school. *The New Zealand Mathematics Magazine*, 37(2), 11-21.
- Higgins, J. (2005). Effective teaching strategies for Māori students in an English-medium numeracy classroom. In J. Higgins, K.C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand numeracy development project 2004* (pp. 74-79). Wellington: Learning Media.
- Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Hill, J., & Hawk, K. (2000). *Making a difference in the classroom: Effective teaching practice in low decile, multicultural schools*. Wellington: Ministry of Education Research Division.
- Holton, D., & Thomas, G. (2001). Mathematical interactions and their influence on learning. In D. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (pp. 75-104). Dordrecht, the Netherlands: Kluwer.
- Horowitz, F., Darling-Hammond, L., & Bransford, J. (2005). Educating teachers for developmentally appropriate practice. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world* (pp. 88-125). San Francisco: John Wiley & Sons.
- Hoyle, C. (1982). The pupil's view of mathematics learning. *Educational Studies in Mathematics*, 13, 340-372.
- Hunter, R. (2005). Reforming communication in the classroom: One teacher's journey of change. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonagh, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 451-458). Sydney: MERGA.
- Hyett, M. (2005). *Interactions and feedback with diverse students in mathematics classrooms*. Unpublished master's project, Massey University, Palmerston North, New Zealand.

- Irwin, K.C. (2005). *Evaluation of a teacher professional development project: 2004 Effective Numeracy Teaching in Years 1 to 6 Pilot Project (Pasifika Focus)*. Report to Ministry of Education, Wellington.
- Jaworski, B. (1994). *Investigating mathematics teaching*. London: Falmer Press.
- Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In M.J. Hoines & A.B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp.17-36). Bergen: PME.
- Jones, A. (1986). "At school I've got a chance..." *Social reproduction in a New Zealand secondary school*. Unpublished doctoral dissertation. Auckland University, New Zealand.
- Jones, A. (1991). "At school I've got a chance": *Culture/privilege: Pacific Islands and Pakeha girls at schools*. Palmerston North: Dunmore Press.
- Khisty, L.L. (1995). Making inequality: Issues of language and meaning in mathematics teaching with Hispanic students. In W. Secada, E. Fennema, & L.B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 279-287). Cambridge: Cambridge University Press.
- Khisty, L.L., & Chval, K.B. (2002). Pedagogic discourse and equity in mathematics: When teachers talk matters. *Mathematics Education Research Journal*, 14(3), 154-168.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Knight, N. (2003). Teacher feedback to students in numeracy lessons: Are students getting good value? *SET* 3, 40-45.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29-63.
- Lampert, M., & Blunk, M. (Eds.) (1998). *Talking mathematics in school: Studies of teaching and learning*. New York: Cambridge University press.
- Latu, V. F. (2005). Language factors that affect mathematics teaching and learning of Pasifika students. In P. Clarkson, A. Downton, D. Gronn, M. Horne, & A. McDonough (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 483-490). Sydney: MERGA.
- Lerman, S. (1993). The role of the teacher in children's learning of mathematics. In A. J. Bishop, K. Hart, S. Lerman, & T. Nunes (Eds.), *Significant influences on children's learning of mathematics* (pp. 61-99). Paris: UNESCO.
- Lin, F.-L., & Cooney, T.J. (2001). *Making sense of mathematics teacher education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Lobato, J., Clarke, D., & Ellis, A.B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, 36(2), 101-136.
- Lubienski, S. T. (2002). Research, reform, and equity in US mathematics education. *Mathematical Thinking and Learning*, 4(2&3), 103-125.
- Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Macfarlane, A.H. (2004). *Kia hiwa ra! Listen to culture: Māori students' plea to educators*. Wellington: NZCER.
- Maddern, S., & Court, R. (1989). *Improving mathematics practice and classroom teaching*. Nottingham: The Shell Centre for Mathematics Education.
- Manouchehri, A., & Enderson, M.C. (1999). Promoting mathematical discourse: Learning from classroom examples. *Mathematics Teaching in the Middle School*, 4(4), 216-222.
- Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum Associates.
- McChesney, J. (2005, April). *Student-initiated interactions: Student agency in mathematics classrooms*. Paper presented at the annual meeting of the American Education Research Association as part of the NZSARE symposium titled: Making meaning: Students agency in four curriculum contexts, Montreal, Canada.
- McClain, K., & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, 32(3), 236-266.
- McMahon, B. (2000). Scaffolding: A suitable teaching characteristic in one-to-one teaching in maths recovery. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 417-423). Sydney: MERGA.
- Meaney, T. (2005). The use of algebra in senior high school students' justifications. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonagh, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 537-544). Sydney: MERGA.
- Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Clevedon: Multilingual Matters.
- Mewborn, D. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education*, 30, 316-341.
- Meyer, D.K., & Turner, J.C. (2002). Using instructional discourse analysis to study the scaffolding of student self-regulation. *Educational Psychologist*, 37(1), 17-25.

- Ministry of Education. (2000). *Tukanga Whakatu Kura Kaupapa Māori: Preparation and assessment for new Kura Kaupapa Māori applicants*. Wellington: Author.
- Ministry of Education. (2001). *Curriculum update No. 45: The numeracy story*. Wellington: Learning Media.
- Morrone, A.S., Harkness, S.S., D'Ambrosio, B., & Caulfield, R. (2004). Patterns of instructional discourse that promote the perception of mastery goals in a social constructivist mathematics course. *Education Studies in Mathematics*, 56, 19-38.
- Moschkovich, J. (1999). Supporting the participation of English language learners in mathematical discussions. *For the Learning of Mathematics*, 19, 11-19.
- Myhill, D., & Warren, P. (2005). Scaffolds or straitjackets: Critical moments in classroom discourse. *Educational Review*, 57(1), 55-69.
- Nash, R., & Harker, R. (2002). How are school composition effects and peer group mechanisms related? A theoretical and methodological discussion from the *Progress at School Project*. *New Zealand Journal of Educational Studies*, 37(2), 171-190.
- Nathan, M.J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction*, 21(2), 175-207.
- Nathan, M.J., & Koedinger, K. R. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education*, 31(2), 168-190.
- Neville-Barton, P., & Barton, B. (2005). *The relationship between English-language and mathematics learning for non-native speakers* (Final Report). Retrieved 7 September 2005, <http://www.tlri.org.nz/index.html>
- Noddings, N. (1995). *Philosophy of education*. Oxford: Westview Press.
- O'Connor, M.C. (1998). Language socialisation in the mathematics classroom: Discourse practices and mathematical thinking. In M. Lampert & M. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 17-55). Cambridge: Cambridge Press.
- O'Connor, M.C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, 46, 143-185.
- O'Connor, M.C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Eds.), *Discourse, learning and schooling* (pp. 63-103). New York: Cambridge University Press.
- Palmer, P.J. (1998). *The courage to teach: Exploring the inner landscape of a teacher's life*. San Francisco: Jossey-Bass.
- Perso, T. (2003). School mathematics and its impact on cultural diversity. *Australian Mathematics Teacher*, 59(2), 10-16.
- Peter-Koop, A. (2002). Real-world problem solving in small groups: Interaction patterns of third and fourth graders. In B. Barton, K.C. Irwin, M. Pfannkuch, & M.O. J. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 559-566). Sydney: MERGA.
- Planas, N., & Gorgorió, N. (2004). Are different students expected to learn norms differently in the mathematics classroom? *Mathematics Education Research Journal*, 16(1), 19-40.
- Popkewitz, T.S. (1988). Institutional issues in the study of school mathematics: Curriculum research. In A.J. Bishop (Ed.), *Mathematics education and culture* (pp. 221-249). Dordrecht: Kluwer.
- Ramaprasad, A. (1993). On the definition of feedback. *Behavioral Science*, 28, 4-13.
- Rawlins, P. (in progress). *Students' perceptions of the formative potential of NCEA*. Unpublished PhD thesis, Massey University, Palmerston North.
- Ruthven, K. (2002). Assessment in mathematics education. In L. Haggarty (Ed.), *Teaching mathematics in secondary schools* (pp. 176-191). London: Routledge Falmer.
- Sadler, R. (1989). Formative assessment and the design of instructional systems. *Instructional Science*, 18, 119-144.
- Schifter, D. (2001). Learning to see the invisible. In T. Wood, B. Scott-Nelson & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109-134). Mahwah, NJ: Lawrence Erlbaum Associates.
- Schifter, D., & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.
- Scribner, S., & Cole, M. (1981). *The psychology of literacy*. Cambridge, MA: Harvard University Press.
- Setati, M., & Adler, J. (2001). Code-switching in a senior primary class of secondary-language mathematics learners. *For the Learning of Mathematics*, 18, 34-42.
- Sfard, A., & Keiran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8(1), 42-76.
- Sherin, M. G. (2002). When teaching becomes learning. *Cognition and construction*, 20(2), 119-150.
- Shulman, L., & Shulman, J. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies*, 36(2), 257-271.

- Slavin, R. E. (1990). Achievement effects on ability grouping in secondary schools: A best-evidence synthesis. *Review of Educational Research*, 60(3), 471-499.
- Slavin, R. E. (1995). *Cooperative learning: Theory, research, and practice* (2nd Ed.). Boston: Allyn & Bacon.
- Springer, L., Stanne, M.E., Donovan, S.S. (1997). *Effects of small-group learning in undergraduate in science, mathematics, engineering, and technology: A meta-analysis*. Research Monograph No. 11. National Institute for Science Education, University of Wisconsin-Madison.
- Steinberg, R. M., Empson, S. B., & Carpenter, T. P. (2004). Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education*, 7, 237-267.
- Stigler, J.W. (1988). Research into practice: The use of verbal explanation in Japanese and American classrooms. *Arithmetic Teacher*, 36(2), 27-29.
- Stipek, D., Salmon, J. M., Givvin, K. B., Kazemi, E., Saxe, G., & MacGyvers, V. L. (1998). The value (and convergence) of practices suggested by motivation research and promoted by mathematics education reformers. *Journal for Research in Mathematics Education*, 29, 465-488.
- Sullivan, P., Mousley, J., & Zevenbergen, R. (2003). The context of mathematics tasks and the context of the classroom: Are we including all students? *Mathematics Education Research Journal*, 15(2), 107-121.
- Taylor, J., & Cox, B. D. (1997). Microgenetic analysis of group-based solution of complex two-step mathematical word problems by fourth graders. *Journal of the Learning Sciences*, 6(2), 183-226.
- Thomas, G. (1994). *Discussion in junior mathematics: Helping one another learn?* Unpublished doctoral dissertation, University of Otago, Dunedin, New Zealand.
- Thomas, G, Tagg, A., & Ward, J. (2002). Making a difference: The Early Numeracy Project. In B. Barton, K. Irwin, & M. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematic Education Research Group of Australasia, Vol. 1, pp. 49-57). Sydney: MERGA
- Thomas, G., & Ward, J. (2002). *An evaluation of the Early Numeracy Project 2001: Exploring Issues in Mathematics Education*. Wellington: Ministry of Education.
- Thornton, C.A., Langrall, C. W., & Jones, G.A. (1997). Mathematics instruction for elementary students with learning disabilities. *Journal of Learning Disabilities*, 30(2), 142-150.
- Trinick, T. (2005). Te Poutama Tau: A case study of two schools. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand numeracy development project 2004* (pp. 80-88). Wellington: Learning Media.
- Turner, J. C., Meyer, D. K., Anderman, E. M., Midgley, C., Gheen, M., Yongjin, K., & Patrick, H. (2002). The classroom environment and students' reports of avoidance strategies in mathematics: A multimethod study. *Journal of Educational Psychology* 94(1), 88-106.
- Turner, J. C., Meyer, D. K., Cox, K. E., Logan, C., DiCinto, M., & Thomas, C. T. (1998). Creating contexts for involvement in mathematics. *Journal of Educational Psychology*, 90(4), 730-745.
- Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London: Routledge.
- Walshaw, M. (1999). *Paradox, partiality and promise: A politics for girls in mathematics*. Unpublished doctoral dissertation, Massey University, Palmerston North, New Zealand.
- Walshaw, M. (2004). The pedagogical encounter in postmodern times: Learning from Lacan. In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 121-140) Greenwich, CT: Information Age.
- Walshaw, M. (2005). Getting political and unravelling layers of gendered mathematical identifications. *Cambridge Journal of Education*, 35(1), 19-34.
- Warfield, J. (2001). Where mathematics content knowledge matters. In T. Wood, B. Scott-Nelson & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 135-155). Mahwah, NJ: Lawrence Erlbaum Associates.
- Watson, A. (2002). Instances of mathematical thinking among low attaining students in an ordinary secondary classroom. *Journal of Mathematical Behavior*, 20, 461-475.
- Webb, N.M. (1991). Task-related verbal interaction and mathematics learning in small groups. *Journal for Research in Mathematics Education*, 22, 366-389.
- Wenger, J. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.
- White, D. Y. (2003). Promoting productive mathematical classroom discourse with diverse students. *Journal of Mathematical Behaviour*, 22, 37-53.
- Whitenack, J.W., Knipping, N., & Kim, O-K. (2001). The teacher's and students' important roles in sustaining and enabling classroom mathematical practices: A case for realistic mathematics education. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 415-422). Utrecht: PME.
- William, D. (1999). Formative assessment in mathematics: Part 2: Feedback. *Equals*, 5(3), 8-11.
- Wood, T. (2001). Learning to teach mathematics differently: Reflection matters. In M.V. Heuvel-Panhizen (Ed.), *Proceedings of the 25th conference of the International group for the Psychology of Mathematics Education* (pp. 431-438). Utrecht: PME.

- Wood, T. (2002). What does it mean to teach mathematics differently? In B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australia, Vol. 1, pp. 61-67) Sydney: MERGA
- Wood, T., & Yackel, E. (1990). The development of collaborative dialogue in small group interactions. In L.P. Steffe & T. Wood (Eds.), *Transforming early childhood mathematics education: An international perspective* (pp. 244-253). Hillsdale, NJ: Erlbaum.
- Woodward, J., & Irwin, K. (2005). Language appropriate for the New Zealand Numeracy Project. In P. Clarkson, A. Downton, D. Gronn, M. Horne, & A. McDonough (Eds.), *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 799-806). Sydney: MERGA.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 2(4), 458-477.
- Yackel, E., Cobb, P., & Wood, T. (1990). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behavior*, 11(4), 469-488.
- Young-Loveridge, J., Taylor, M., & Hawera, N. (2005). Going public: Students' views about the importance of communicating their mathematical thinking and solution strategies. *Findings from the New Zealand Numeracy Development Project 2004*. Wellington: Learning Media.
- Zack, V., & Graves, B. (2002). Making mathematical meaning through dialogue: "Once you think of it, the z minus three seems pretty weird. *Educational Studies in Mathematics*, 46, 229-271.

5. *Mathematical Tasks, Activities and Tools*

Introduction

Within classrooms involving mathematical communities of practice, teachers need support to achieve their mathematical agenda. This chapter discusses how this support manifests itself in the form of instructional tasks and the tools available for solving those tasks.

Research in New Zealand is increasingly showing that task design plays a central role in structuring and developing an effective learning community. The social and the cognitive are not distinct domains in practice, but are integrated and embedded in task and activity design and classroom organisation. (Alton-Lee, 2003, p. 27)

In the mathematics classroom, it is through tasks, more than any other way, that opportunities to learn are made available to students. Tasks are defined by Doyle (1983) as the “products that students are expected to produce, the operations that students are expected to use to generate those products, and the resources available to students while they are generating the products” (p. 161). Tasks and their associated activity are hugely significant in determining how students come to understand mathematics (Doyle, 1988). The mathematical tasks with which learners engage determine not only what substance they learn, but how they come to think about, develop, use, and make sense of mathematics:

the cumulative effect of students’ experience with instructional tasks is students’ implicit development of ideas about the nature of mathematics—about whether mathematics is something they personally can make sense of, and how long and how hard they should have to work to do so. (Stein, Smith, Henningsen, & Silver, 2000, p. 11)

In considering tasks, we look first at how the cognitive activity associated with mathematical thinking can be supported by task purpose and design. Second, we consider the relationship of the task to the learner, providing evidence of the importance of teacher knowledge and expectations in ‘hearing’ the competencies of their students and building from them. We also consider how tasks are mediated by pedagogical affordances and constraints and the participation norms of the classroom. Third, we provide evidence of how teachers can support the use of tools as resources or learning supports. We see that it is not the tool (or the inscription) in isolation that offers support for the teacher; rather, it is the learners’ use of the tool and the meanings that develop as a result of this activity. In this way, the tool is not seen as standing apart from the activity of the learner. As in previous chapters, we acknowledge the critical role of teacher knowledge in the complex instructional dynamic.

Tasks that are problematic and have a mathematical focus provide opportunities for mathematical thinking

Mediated by communities of practice and related mathematical and social norms, tasks introduce important mathematical ideas and provide opportunities for learners to engage in a range of thinking practices (Marton & Runesson, 2004). At every level, this requires “the development of knowledge of concepts, techniques, notations, and relationships; recognising them in familiar and unfamiliar forms; recalling facts, names, procedures; using procedures fluently and accurately; the ability to shift between methods and representations; applying knowledge to solve problems, possibly transforming it to do so; and creating generalisation, abstractions, images and methods” (Watson, 2004, p. 364).

Given the range of cognitive and metacognitive demands within the various strands of mathematics—for example, mastering early computational procedures, generalising number structure, visualising three-dimensional shapes, interpreting statistical data, imagining limits

in calculus, and dealing with complex numbers—tasks vary in format and purpose. Tasks that require students to engage in complex and non-algorithmic thinking promote exploration of connections across mathematical concepts (Stein, Grover, & Henningsen, 1996); tasks that require students to model their thinking promote reflection (Fraivillig, Murphy, & Fuson, 1999); tasks that require students to discern invariants and variation, and structure, promote generalisation (Watson & Mason, 2005); tasks that require students to interpret and critique data promote the disposition of ‘scepticism’ (Chatfield, 1998); tasks that require students to ‘notice and wonder’ promote the disposition of curiosity (Shaughnessy, 1997); and tasks that provide opportunities for ‘mathematical play’ promote conjecture and exploration (Holton, Ahmed, Williams, & Hill, 2001). However, according to Hiebert et al. (1996), tasks should share some commonality: they should be problematic for the learner and leave a mathematical ‘learning residue’ (Davis, 1992). Most importantly, this residue should consist of (a) insight into the structure of mathematics and (b) strategies or methods for solving problems.

Students who engage in meaningful mathematical tasks are potentially able to treat situations as problematic: something they need to think about, not simply a disguised way of practising already-demonstrated algorithms. To engage in problem-based tasks, students must impose meaning and structure, make decisions about what to do and how to do it, and interpret the reasonableness of their actions and solutions (Holton, Spicer, Thomas, & Young, 1996). Utilising tasks that have more than one solution strategy and which can be represented in multiple ways requires that students communicate and justify their procedures and understanding (Carpenter, Franke, & Levi, 2003). Tasks that fit these criteria “leave behind something of mathematical value” to the learner (Hiebert et al., 1997).

The New Zealand Numeracy Development Project (NDP) advocates problem-based tasks that focus on students’ sense-making activities. Evaluation reports suggest that tasks that require students to justify their solution strategies and reflect on their thinking support student gains in computational proficiency. Teachers have given such tasks credit for creating more positive learning environments (Higgins, Bonne, & Fraser, 2004; Thomas & Tagg, 2004; Thomas & Ward, 2002). From an action research study involving two year 9 classes, Holton, Anderson, Thomas, and Fletcher (1999) report considerable improvement in the performance of a class of traditionally low-attaining students as a result of increased opportunities to engage in problem-solving activities.

Large-scale empirical studies of educational change in the US also link significant achievement gains to changes in classroom practices centred on enquiry-based problem-solving approaches (e.g., Swanson & Stevenson, 2002; Thomas & Senk, 2001). Balfanz, MacIver, and Byrnes (2006) researched the implementation of the Talent Development (TD) Middle School mathematics programme in high-poverty schools. The programme combined evidence-based mathematics reforms with a focus on problem-solving skills as opposed to routine mathematical procedures. Tracking the first four years, the study found that across all levels of the achievement spectrum, students from the TD classes outperformed control schools on multiple measures of achievement. The average effect size by the end of middle school was .24.¹

One of the significant features of these large-scale reform programmes has been teacher professional development with a focus on curriculum development and planning. Stein et al.’s (1996) analysis of a sample of 144 tasks within the QUASAR middle-school programme found that nearly three-quarters of the tasks required students to engage in high-level cognitive processes—either the active “doing of mathematics” (40%) or the use of procedures with connections to concepts, meaning, or understanding (34%). Eighteen percent of the tasks focused on the use of procedures without making connections to concepts, meaning, or understanding. These tasks possibly fulfilled the intention of practice and consolidation—a necessary feature of mathematics learning (Anthony & Knight, 1999b). Overall, the researchers concluded that the tasks embodied many of the characteristics that support students’ capacity to think and reason about mathematics in complex ways. This conclusion was strengthened by Stein’s (2001) review of a range of studies that provided evidence that the move to a problem-

solving approach in mathematics teaching has been associated with “increases in student performance on assessment tasks that measure students’ capacity to think, reason and communicate” (p. 112).

While research has shown that quality teaching focuses on the mathematical aspect of the task (Blanton & Kaput, 2005), teachers’ attempts to make mathematics interesting are sometimes at the expense of accuracy and meaning (Christensen, 2004). Rubick’s (2000) research into a statistical investigation carried out by a group of New Zealand year 7 and 8 students notes that the students were able to select an investigation (e.g., eye colour) that focused on counting data sets rather than the intended exploration of relationships within data. The teachers of middle grades in Moyer’s (2002) study reported that they often used manipulatives to ‘have fun’, an activity which they distinguished from their ‘real maths’ instruction. The tension between the goal of creating a positive climate in the classroom and the need to foster mathematical understanding is illustrated in the following profile of Linda Arieto, a Puerto Rican teacher who shares a cultural and linguistic background with her students.

Linda’s Lesson

Linda’s lesson [on graphing] challenged traditional approaches to mathematics instruction in a number of ways. First, her lesson involved statistics ... Second, the nature of the graphing task centered on a real-world application of mathematics: collecting, organizing, and describing actual data. Finally, this lesson actively involved students in working collaboratively. Students worked in pairs and were thoroughly engaged in the task ... When asked about the purposes of the matchbook graphing lesson, Linda explained she wanted to “just expose them to graphing.” Her ongoing goal was for students to “develop a tremendous love for maths,” a love she was unable to cultivate for herself. Perhaps for this reason Arieto steered away from emphasising accuracy and validity in the graphing work ... Her emphasis on “having fun” was in keeping with her goal that students enjoy school and therefore continue to attend in a city region where dropout rates can be as high as 85% by the high school years. However, Linda seemed to miss the potential to capture students’ interest by using mathematical tasks that fostered important conceptual learning, maintained high standards, and were challenging and engaging.

From Cahnmann and Remillard (2002)

Other studies report similar tensions. Bills and Husbands (2005) report on the practice of a secondary school mathematics teacher’s efforts to consciously negotiate between her espoused values within mathematics education and education more generally. A tension existed between her desire to develop her students’ sense of social well-being alongside their sense of mathematical well-being. As she perceived it, her fundamental role was to develop students’ confidence and give them a sense of what they were capable of achieving. However, a pedagogy focused principally on the development of a non-threatening learning climate does not get to the heart of what mathematics teaching really entails (Connor & Michaels, 1996).

The organisation of social interaction within group settings can also override the mathematical focus of a task. Higgins (1998) found that group settings provided by New Zealand junior primary school teachers frequently did not involve carefully structured tasks, nor did they have classroom norms that supported student engagement with mathematical ideas or concepts. The following episode from Higgins’ research illustrates the tension between a teacher’s need to manage her class and her desire to promote learning opportunities within the context of a group task.

The Jigsaw Puzzle

The teacher at Sapphire School describes the use of independent tasks and group work in terms of an organisational device: *one where you don’t need to sit down and teach them too much ... because*

it's pretty self-explanatory. Because often you don't get time to sit down and teach the independent activities separately from the others.

In describing an episode of an independent group activity involving jigsaw completion, Higgins notes that two six-year-old girls, Carol and Suzanne, started out by randomly picking up the loose pieces and trying to fit them together before placing them on the board. The muted colour of the pieces provide little clue as to the position of the pieces. After a short while Suzanne remarked, "Maybe a bit hard for us". The girls then tried to work from a corner piece but after 5 minutes on the puzzle, with no solution in sight, the girls played with the pieces, laughing and singing as they placed the pieces in the wrong place. After an interval of off-task behaviour, the girls unsuccessfully attempted to solicit some peer help. Another peer, Catherine, was approached. While they did get much of the puzzle done with Catherine's help, this was a less than positive experience. Catherine derided Carol and Suzanne's efforts: "*It's the other way derbrain,*" and proceed to take over much of the completion of the puzzle.

Higgins' conclusion that Suzanne and Carol were unlikely to have learned any new strategies for jigsaw puzzle solving through this experience was in direct contrast to the perceptions of the teacher. Her reflection of the episode, during which she was teaching another group was: *[They] sat down and did it and they really discussed it and they really tried to solve it and they—they worked really hard on it and they finished it and it took them a LONG time but they worked really well together. ... it's good to see that sort of thing happening where the children ... do a mistake and they actually have to work out what's gone wrong ... it's good to see them make mistakes and then work it out.*

Higgins suggests that the over-reliance on working with one small teacher-led group can potentially result in lost opportunities for mathematical learning. The jigsaw puzzle activity was intended to support the students' knowledge and skills about mathematical relations. However, apart from the mention of the concept of "fit," the puzzle appeared to be interpreted by the girls, and by a peer, as a measure of competence. In order to focus the students on the mathematical nature of the activities Higgins argues for a greater use of whole-class introductions, including discussion of possible strategies, negotiated ways in which children can assist each other, and plenary sessions which involve student and teacher reflection.

From Higgins (1997)

Likewise, Stein (2001) reports instances of displaced learning in co-operative tasks that have been insufficiently structured to engage students with mathematical ideas. Without an explicit focus on the mathematical structure and processes, mathematics learning may be incidental. Instead of marginalising tasks as mathematical 'field trips' or enrichment activities which occur in isolation, Blanton and Kaput (2005) argue that teachers must expect mathematical thinking to be "woven into the daily fabric of instruction" (p. 440).

The importance of attending to the structure of mathematics is emphasised in the influential study, *Effective Teachers of Numeracy* (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997). From this study, Askew and colleagues conclude that highly effective teachers—referred to as *connectionist* teachers—consistently attend to connections between different aspects of mathematics, for example addition *and* subtraction or fractions, decimals, *and* percentages.

I think you've got to know that they are inverse operations. Those two (addition and subtraction), and those two (multiplication and division) are linked, because when you are solving problems mentally you are all the time making links between multiplication, division, addition and subtraction. (Barbara, in Askew, 1999, p. 99)

Mulligan, Mitchelmore, and Prescott's (2005) research examined 103 first grade students' representations as they solved 30 tasks across a range of mathematical content domains such as counting, partitioning, patterning, measurement, and space.

Patterns

Mulligan and colleagues found that early school mathematics achievement was strongly linked with the child's development and perception of mathematical structure. Low achievers focused on superficial characteristics of problems; they consistently showed lack of attention to the mathematical or spatial structure. In contrast, high achievers were able to draw out and extend structural features, and demonstrated strong relational understanding in their responses. Figures 5.1 and 5.2 show two responses to a triangular pattern task in which the pattern was reconstructed from memory and extended, and illustrate clear differences in the development of the two children concerned over an 18-month period. In comparison to the high-achieving child's advances in structural representations, the low-achieving child demonstrates only partial awareness of patterns, and no developmental growth in structure.

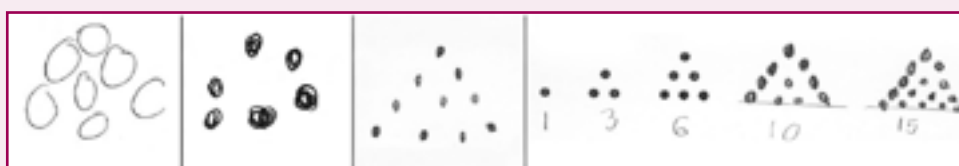


Fig. 5.1. Structural stage development of high-achieving child



Fig 5.2. Structural stage development of low-achieving child

The researchers argue that low achievers may benefit from access to tasks that assist them in visual memory and recognition of basic mathematical and spatial structure in objects, representations and contexts. Assisting children to visualise and record simple spatial patterns accurately could potentially lead to much broader improvements in children's mathematical understanding.²

From Mulligan, Mitchelmore, and Prescott (2005)

Research studies have found that children's exploration of mathematical structure within early numeracy provides an effective bridge from numerical to early algebraic thinking. Carpenter, Franke, and Levi's (2003) research presents classroom episodes in which young children productively explore the interface between arithmetic and algebra. The focus of their research is the use of conjecture, or claims that seem plausible but which are not yet established (e.g., when you add an odd number to another odd number, the answer is an even number). In the New Zealand context, Irwin and Britt (2005) found that students in the NDP solved numerical problems that required manipulation more successfully than did students who had not participated in the project. The development of a "flexible array of skills for manipulating arithmetical relations in ways that exhibit number sense as well as operational sense where students develop understanding of flexible numerical structures involving the four arithmetic operations" (Irwin & Britt, p. 182) appears to facilitate the acquisition of algebraic thinking. While this study reported a significant socio-economic status (SES) effect, another study (Irwin & Britt, 2004), involving two low-decile schools and one middle-decile school, reported no SES effect on student achievement.

In a year-long investigation into the characteristics of instructional practice that support the development of algebraic reasoning, a third grade teacher's practices were analysed. The following vignette illustrates how the teacher, June, instead of implementing pre-designed algebraic tasks as stand-alone activities, incorporated opportunities for algebraic reasoning³ into regularly planned—and what may have been more arithmetically focused—instruction.

Algebraic Treatment of Number

In the following episode June treated numbers in an algebraic way, that is, as a placeholder that required students to attend to structure rather than rely on the computation of specific numbers. In the interchange, June challenges a student's use of an arithmetic strategy to deduce that $5 + 7$ was even:

- June: How did you get that?
- Tony: I added 5 and 7 and then I looked over there [pointing to a visible list of even and odd numbers on the wall] and saw that it was even.
- June: What about $45678 + 85631$? Odd or even?
- Jenna: Odd.
- June: Why?
- Jenna: Because 8 and 1 is even and odd, and even and odd is odd.

The introduction of large numbers required the students to think in terms of even and odd properties to determine parity. In doing so, the researchers maintain that June used numbers as placeholders, or variables, for any odd or even numbers. Moreover, the researchers noted that in using numbers algebraically the teacher was able to avoid the semiotic complications of using literals (e.g., $2n + 1$ for some integer n) to represent arbitrary even and odd numbers. This illustrates how the abstractness of numbers gets built as students work with particular quantities, and how the teacher can set the stage for the next move—the formal expression of the generalisation.

From Blanton and Kaput (2005)

In the secondary school context, researchers from the Improving Attainment in Mathematics Project (IAMP) attributed improvements in students' mathematical attainment to teachers and learners focusing on the development of ways to think with, and about, key ideas in mathematics. Like researchers in the primary context, De Geest, Watson, and Prestage (2003) note the importance of connections within mathematics:

The 'contents of the subject-matter domain' are deeply connected within themselves through mathematical structure, and that enculturation into mathematical thinking involves becoming fluent with constructing, creating and navigating similar or isomorphic structures, that is, being intimately attuned to the ways in which mathematics is internally connected. (p. 306)

Watson's (2002) experiences with low attainers convinced her that "some potentially powerful mathematical talents of these students were unrecognised and unused in the teaching of mathematics" (p. 472). In advocating the benefits of pedagogy focused on awareness of mathematical structure, Watson provides the following example to illustrate these students' engagement in mathematical thinking.

Mathematical Talent

Secondary students from a low-attaining class grouping had been using flow diagrams to calculate the outputs of compound functions such as shown in figure 5.3.

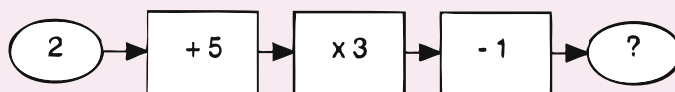


Fig. 5.3. A flow diagram

In response to a request to make up some hard examples of their own, most students provided examples with more operations and bigger numbers. However, one student suggested constructing problems in which the operations and output are known and the input has to be found. Another student provided an example in which input and output were given but the last operation was missing. According to Watson, these two students were working with the relations rather than the numbers and operations. They saw the structure of the problem as something they could vary, rather than following the template of teacher-given questions. The generation of the special examples enabled the students to shift to more complex levels of mathematical thinking.

From Watson (2002)

Watson (2002) readily acknowledges the difficulties of working with low-attaining students. However, as a result of her research findings, she offers three principles for working with low attainers, based on student task construction and interactive strategies that are focused on mathematical thinking:

1. Their attention can be drawn to structure through observing patterns which go across the grain of work.
2. They can be asked to exemplify, and hence get a sense of structure, generality, and extent of possibilities.
3. They can be prompted to articulate similarities in their work, and hence be prompted to represent similarities in symbols.

Confirming the importance of mathematical focus, Watson (2003) claims that teachers can learn much about students' responses to tasks by thinking about what the task affords in terms of activity, what is constrained, and what attunements or patterns of participation are brought to bear in the activity that is generated by the task. Watson, along with other researchers (e.g., Marton & Tsui, 2004; Runesson, 2005), suggests that teachers can focus and refine the opportunities for learning mathematics by controlling the amount of variation permitted in any task or series of tasks. While each mathematical task affords opportunities to learn, it is by limiting variation to the feature on which it is hoped students will focus and by inviting conjecture and generalisation that students can be directed to the construction of mathematical meaning.

Marzano, Pickering, and Pollock (2001) analysed numerous classroom-based research studies. They suggest that explicit guidance in identification of similarities and differences, accompanied with opportunities for students to independently identify similarities and differences, "enhances students' understanding of and ability to use knowledge" (p. 15). To illustrate the power of task variation, Runesson (2005) provides the following vignette (cited in Jaworski, 1994) that involves an exchange between two girls working on a task relating to volume and surface area.

Claire's Task Variation

In the episode, the teacher, Claire, has set up a provocative situation by challenging her students with an apparent contradiction.

- Cl: We're saying, volume, surface area and shape, three sorts of variables, *variables*. And you're saying, you've fixed the shape—it's a *cuboid*. And I am going to say to you, hm.
- Cl: I'll be back in a minute.
- Cl: *That's* a cuboid. [She picks up a tea packet]. *That* is a cuboid. [She picks up an electric bulb packet.] [She picks up a meter ruler.] *This* is a cuboid. [She looks around their faces. Some are grinning.] And you are telling me that those are all the same shape? [Everyone grins.]
- R: Well, no-o. They've all got six separate sides though.
- Cl: They've all got six sides. But I wouldn't say that that is the same shape as that. [She compared the meter rule with the bulb box.]
- R: No-o.
- Cl: Why not?
- D: Yes you would ... [There is an inaudible exchange between the girls D and R.]
- Cl: What is different? [Hard to hear responses include the words *size* and *longer*.]
- Cl: Different in *size*, yes. [Clare reached out for yet another box, a large cereal packet, which she held alongside the small cereal packet.] Would you say that *those* two are different shapes?
- R: They're similar.
- Cl: What does similar mean?
- R: Same shape, different sizes.

[During the last four exchanges there was hesitancy, a lot of eye contact, giggles, each person looking at others in the group, the teacher seeming to monitor the energy in the group.]

- Cl: Same shape but different sizes. That's going around in circles isn't it?—We still don't know what you mean by *shape*. What do you mean by shape?

[She gathers three objects, the two cereal packets and the meter ruler. She places the rule alongside the small cereal packet.]

- Cl: This and this are different shapes, but they're both cuboids.

[She now puts the cereal packets side by side.]

- Cl: This and this are the same shape and different sizes. What makes them the *same* shape?

[One girl refers to a scaled-down version. Another to measuring the sides—to see if they're in the same ratio. Claire picks up their words and emphasises them.]

- Cl: Right. So it's about *ratio* and about *scale*. (pp. 74–75)

Runesson suggests that the mathematical challenge presented to these students involves the discernment of similarities and differences between the packages to determine what 'the same shape' means? To assist students in this task, the teacher brings out the similarities and differences between various 3D objects:

Initially, the teacher kept the number of sides of the shapes constant, whereas the lengths of the sides varied; the lengths, breadths and the heights were not proportional ... the opening of a space for variation enabled two learners to discern those particular aspects of the object of learning. (pp. 76–77)

From Runesson (2005)

By contrasting the task focus of two lessons, Watson (2003) also illustrates how task variation can impact on students' opportunities to engage in mathematical thinking and learning. In the first lesson (see Groves & Doig, 2002), the mathematics focus was the 'concept of a circle'. The lesson began with Mr J. producing a pole for a game of quoits. The students stood along a straight line and threw rings over the pole, which was a short distance from the line. The class then discussed how easy or hard it was to do this and 'measured' and compared distances, using a tape. In response to the teacher's challenge to "make the game fair" the students conjectured that they should stand in a curve. In a second lesson at a different school, the teacher identified the lesson topic as "The chance of winning one million dollars". The activity involved the students tossing three coins and recording their results. The plenary session consisted of a rather inconclusive discussion about the different results found by the students.

According to Watson's (2003) analysis, the dimension of task variation involved in the first lesson was the distance from the point. This was enacted by students, represented by string, and made the focus for discussion. Groves and Doig (2002) reported that this focus on the mathematical properties of a circle—supported by appropriate attention to social and mathematical norms—appeared to be obvious to all the students. By way of contrast, the task in the second lesson included variations in numbers of trials, methods of throwing coins, ways of recording, and ways of comparing results. While it is likely that any teacher observing this lesson would know that the lesson was about probability, Groves and Doig noted that the mathematical focus would be less obvious for students. Thus, Watson contends that while the second lesson created multiple opportunities to learn, some students would be learning about ways of recording, others about fractions, others about how to work together, others about how to avoid working, and so on, only a few might learn about experimental probability. To increase the effectiveness of this lesson for all students, Watson suggests that as a teacher she would need to:

narrow the range of what it is possible to learn, to discern as varying, in what I offered disparate students and thus increase the opportunity to learn appropriate mathematics for as many students as possible, while making sure that they all had access to the patterns under consideration. (Watson, 2003, pp. 36–37)

A critical focus of task activity is students' solution strategies (Hiebert et al., 1997). The *Effective Teachers of Numeracy* study (Askew et al., 1997) notes that, in addition to valuing mathematical structure, *connectionist* teachers in this project consistently value children's solution strategies. The importance of teachers noticing and attending to the mathematics inherent in their students' solution strategies is highlighted in numerous classroom studies (e.g., Carpenter et al., 1997; Sherin, 2001) and has been discussed in more depth in the preceding chapter. The following episode from a Japanese classroom illustrates how alignment of the mathematical focus of a task with the associated student activity can be effective in developing students' mathematical thinking and understanding.

Adding It Up

In the Year 1 lesson the children's problem for the day was to find the answer to $8 + 6$ and explain the reasons for their answers.

Children worked individually for 5 minutes, after which the teacher wrote $8 + 6 = 14$ on the board and invited particular children to write their solutions on the board.

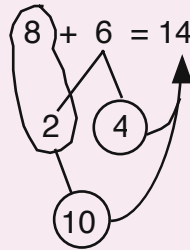


Fig. 5.4. Girl 1's solution for $8 + 6 = 14$

Girl 1's solution is shown in figure 5.4. When asked, most children stated that they had used the same method. The teacher then asked the children to guess why Girl 1 had divided the 6 into 2 and 4. Children responded that this was based on "Nishimoto-san's making 10 rule"—apparently formulated by one of the children, Nishimoto-san, in the previous lesson where the problem was to find $9 + 8$.

The teacher then asked for a different solution. Boy 1's solution is shown in figure 5.5:

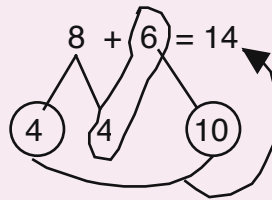


Fig. 5.5. Boy 1's solution for $8 + 6 = 14$

The teacher commented that this was again using "Nishimoto-san's making 10 rule", and asked for another way. Girl 2's solution, still described by the teacher as using "Nishimoto-san's making 10 rule", is shown in figure 5.6. A few children said they had used this method.

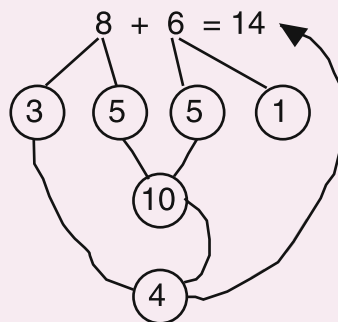


Fig. 5.6. Girl 2's solution for $8 + 6 = 14$

Boy 2 stated that he did not use the "making 10 rule". Children tried to guess how he found the answer—had he used a "making 5 rule"? Boy 2 said he had not and explained his reasoning as shown in figure 5.7:

$$8 + 6 = 14$$

because $9 + 6 = 15$ "we did this before"

and 8 is one less than 9. So, "if 9 becomes 8, the answer is one less".

$$\begin{array}{c}
 8 + 6 = 14 \\
 \swarrow \quad \searrow \\
 1 \text{ less} \rightarrow 9 + 6 = 15 \rightarrow 1 \text{ less}
 \end{array}$$

Fig. 5.7. Boy 2's solution for $8 + 6 = 14$

Many children clapped in response to this solution and a girl commented that this used their former knowledge of addition. The teacher suggested that they move on to looking at $7 + 6$ using the same method [and the lesson continued with students conjecturing and testing hypotheses around the 'making 10' rule].

It is evident that the core task focus—the mathematical structure of the numbers (e.g., $8 = 3 + 5$, $6 = 5 + 1$)—was enhanced by students' conjectures and justifications. The teacher's frequent use of students' solutions, both correct and incorrect, was a feature of the lesson. The lesson purpose was achieved through both the use of a task that was genuinely problematic, yet accessible, for students, and through the establishment of social norms that valued individual and group contributions to the solution process.

From Groves and Doig (2004)

As we have seen in chapter 4, tasks that pivot around mathematical negotiation and sense making afford opportunities for student engagement in mathematical practices. For the teacher, utilisation of mathematics tasks that focus on sense making require them to support students explicitly to take a more responsible role for making mathematical meaning (Anthony & Hunter, 2005; Pape, Bell, & Yetkin, 2003). In classrooms where students actively negotiate meaning, Watson (2003) has found that teachers:

structure the context of such negotiation with examples, counter-examples, or by encouraging the development of these, so that what is eventually learnt is coherent and valid. [In addition] the teacher needs to structure the negotiation process itself so that it is mathematical ... based on exemplification, generalisation, conjecture, justification and so on. (p. 32)

The role of examples

Task focus on mathematics is exemplified by the commonly utilised *worked example*. But far from being a show and tell—followed with practice by replication—the role of examples can be usefully associated with the important process of generality. Watson and Mason (2005) argue that a central aspect of learning mathematics involves "becoming familiar with examples that manifest and illustrate mathematical ideas and by constructing generalisations from examples" (p. 2). Effective instruction should, according to Watson and Mason, include collections of examples—'example spaces'—that exemplify structural similarities and differences. In the following analysis of a task and associated student activity observed in a UK numeracy lesson, Askew (2003) links the students' difficulty in discerning number properties to the inadequate provision of an example space.

Videotapes

The observed task was: *Mrs Chang bought some videotapes. She bought five tapes each costing the same amount. She spent £35. How much did each tape cost?* The essence of this problem is 'what number do I have to multiply by 5 to get 35?' Symbolically: $? \times 5 = 35$.⁴

This problem is difficult to represent in the physical world (using a 'model for') and several of the children used a trial-and-error approach. Using vertical mathematising (that is, working with

the symbols), the model can be recast as $5 \times ? = 35$. The students did not do this. Although they might 'know' that $? \times 5 = 5 \times ?$, they appeared to find it difficult to 'uncouple' from the real-world context and move around the mathematical world instead; to move from a 'model for' to a 'model of' (Gravemeijer, 1994).

Askew argues that task selection should focus on the development of analogical thinking, enabling students to think about the structure of problems. "Rather than treat each problem afresh, the experienced problem solver has knowledge of a wealth of problems, some of which provide generic 'archetypes' that can be used to decide what category of problems a specific example fits into" (p. 84). In this case, the problem could usefully be paired with the problem: *Mr Chang bought some video tapes. He bought some tapes costing £7 each. He spent £42. How many tapes did he buy?* By exploring why one or the other may be more easily solved, children's insights into the nature of problem could be deepened.

From Askew (2003)

The work of the Cognitively Guided Instruction Project (Carpenter et al., 1999) is based on a problem-structuring framework. For example, simple addition problems can be categorised either as *change* (a given quantity is increased) or *combined* (two separate quantities are brought together). Both these categories can give rise to further categories of problems, depending on the position of the 'unknown' in the 'story'. Introducing children to the language of 'change', 'combine' and so on may help them to establish a generic set of problems, especially if these are accompanied by contextual images that can form the basis of archetypal problems (Carpenter et al., 1996).

Student-generated examples and questions can form a productive focus for student enquiry (Watson & Mason, 2005). By engaging in a public discussion of student-generated methods based on student-generated examples, the teacher in this vignette provides opportunities for her learners to become more resourceful and flexible.

Farah's Distributive Law

Farah, an experienced and highly qualified teacher, was working with her class of 11-year-olds on their understanding of multiplication. They had been assessed as having average attainment in their previous schools. In United Kingdom parlance they were classified as *middle ability*, which means that on tests they tended to come somewhere in the middle range.

She had been explicit about the use of the distributive law to deal with different place values of the digits used in large numbers. For example, they had been taught, or reminded, through calculator exploration, that 7×65 was the same as $7 \times 60 + 7 \times 5$ or $7 \times 50 + 7 \times 5 + 7 \times 10$ or other such representations. Learners were asked to contribute examples of multidigit multiplications and show the whole class how they would calculate them. Some learners used a traditional approach of dealing with separate digits, such as $37 \times 9 = 30 \times 9 + 7 \times 9$, but others used ad hoc decompositions that suited the specific numbers being multiplied, such as $37 \times 9 = 40 \times 9 - 3 \times 9$.

Farah particularly praised these decompositions. Her aim was for learners to develop both flexible and mental methods for multiplication, as well as to understand distributivity. The examples were shared within the class. Two weeks later she repeated the exercise. There was a significant increase in most learners' use of flexible approaches based on characteristics of the numbers involved, rather than just separating the digits, although there had been no work on this area of mathematics meanwhile. Farah took this to be a sign that some learning had taken place as a result of the emphasis she had placed on interesting decompositions in the earlier lesson.

From Watson and Mason (2005)

In this episode, the development of exemplification is not explicitly forced by teacher-imposed constraints, but by sharing, making other possibilities available, and publicly valuing examples of what the teacher hopes others will be able to do later. By valuing a range of ways of seeing

the mathematics, the teacher encourages student creativity in arithmetic. Watson and Mason claim that this practice shifts the responsibility for learning to the learners, thus helping them to gain ownership of their mathematics.

Asking learners to exemplify aspects of what they have studied encourages them to search through the structure from varying points of view, using a new dimension, and hence see, perhaps, for the first time, what might be there by discerning features and aspects. Thus, learners might find that being asked to exemplify gives them an opportunity to search in unfamiliar ways through what is familiar to get a more complex sense of the range of possibilities in the topics studied. (p. 31)

In a relatively small New Zealand study, Klymchuk (2005) reports improvement in student performance as a result of the explicit use of counter-examples when teaching calculus. Performance improved on questions requiring conceptual understanding but not on questions requiring the application of familiar rules, algorithms, and calculations. Watson and Mason (2005) and Carpenter, Franke, and Levi (2003) note that student-generated counter-examples are natural ways for all learners to argue and explain their mathematical thinking.

This discussion on the use of examples highlights the potential of a task format that is traditionally taken for granted within the mathematics classroom. In arguing for greater use of examples in mathematics lessons, Watson and Mason (2005) go so far as to say “that until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept” (p. 32). Further research within the New Zealand context and at all levels is needed to substantiate their claim that “learning is greatly enhanced when learners are stimulated to construct their own examples” (p. 32).

Open-ended tasks: modelling, investigating, and playing

Another form of task that supports student exploration and thinking is the open-ended task. Tasks or problems are deemed open-ended when learners must engage in additional problem definition and formulation in order to proceed⁵. This ‘openness’ allows for a range of ‘correct’ responses and a range of ways of achieving those responses. Moreover, the openness of the activity fosters some of the more important aspects of learning mathematics, specifically, “investigating, creating, problematising, mathematising, communicating and thinking” (Sullivan, Warren, & White, 1999, p. 250). Open-ended tasks provide the ideal opportunity for ‘mathematical play’—“that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion” (Holton, Ahmed, Williams, & Hill, 2001, p. 403).

In addition to providing learners with the opportunity to engage in a range of mathematical practices, Zevenbergen (2001) notes the pedagogical advantages of such activity. Her study, which involved 114 primary school students, highlights how the diversity of responses and range of representations offered by open-ended problems allow greater scope for (a) teachers to assess student understanding and (b) students to demonstrate what they know. The students in this study were exploring ‘average’, using examples such as: *At the Chevron Island Bridge, the average number of people per car is 2.5. Draw what this might look like if there are 16 cars on the bridge.* Importantly, Zevenbergen’s study highlights potential drawbacks for some students. When solving the task: *My dog weighs about 20 kilograms. How much could she weigh?*, several students struggled with the ambiguity of the word ‘could’: some used a futures perspective, and others, a rounding context. In a subsequent study, *Overcoming Structural Barriers to Mathematics Learning*, Zevenbergen and colleagues found that teachers needed to be more explicit about task purposes and ‘rules’, especially where the language may be interpreted in different ways. We revisit this in the later section on task context.

Several research studies have focused on the role of mathematical modelling activities

involving authentic problem situations, opportunities for model exploration and application, and multifaceted end products (Lesh & Doerr, 2003). In contrast to the traditional focus on arithmetic word problems, English (2004) argues that we “need to design experiences that develop a broad range of future-oriented mathematical abilities and processes. Mathematical modelling, which has traditionally been reserved for the secondary schools, serves as a powerful vehicle for addressing this need” (p. 207). The following vignette illustrates children’s engagement in a mathematical modelling activity.

What Car to Buy?

The sixth grade class teacher introduced the following modelling activity in a whole class format:

Carl and his mother have been out shopping for cars. *Carl wants a car that will be fun to drive around in, gets good gas mileage, but doesn’t cost too much. But Carl’s mother, who is going to help pay for the car, wants him to have a car that is reliable and safe. Your job is to create a list for Carl and a list for his mother showing which cars are the best. Then they will have to decide which one to buy!*

Table of information (abridged from nine entries):

Car Style	Year	Cost	Color	Mileage	L/100 km city	Features	Body
Nissan Silva	1992	10,000	Navy blue	96,000	10	Rear spoiler, power windows, power steering, CD player, alloy wheels, alarm	Coupe
Honda Legend	1993	17,200	Dark green	15,400	12.5	Dual airbags, anti-lock brakes, alarm, cruise control, electric sunroof, power steering, power windows	Sedan

The children worked in small groups to solve the problem. The following episode is from Jasmine’s group:

- Charlotte: I think we should do a process of elimination. (A brief discussion ensued regarding the mother wanting the car to be reliable.)
- Jasmine: ... Maybe first, maybe we should do a process of elimination, so work our way down the list or work our way up.
- Rachel: We have to consider all the factors though.
- Douglas: (Reminding the group that they need to be objective) She (the mother) is helping him. How about we just judge off what the thing says, not by what we think ... Let’s all read through this again and underline all the details that help us.

After a number of minutes spent revisiting the goal and re-interpreting the problem information the group returned to the process of elimination, with considerable argumentation around the suggestion that the most expensive car be eliminated first. Expectations to justify claims were common: “Jasmine, tell us why you think these things. We need to know why you think them.”

The conversation continued with the group reverting to argumentation over cost factors versus leisure features of the cars. Again Douglas reiterated the need to be objective: “we’re not deciding on what we like, we’re deciding on the facts ... we have to look at these factors.” Finally the group devised a rating system based on the most important features: safety, leisure and extras, mileage.

Car Preferences for Carl and his Mum (abridged from nine entries):

Car	Carl	Mother	Total
Nissan Silva	4th	4th	4th
Honda Legend	7th	6th	7th

At the end of the activity, each group of children shared with the class their approaches to working the activities, explained and justified the model they had developed, and then invited feedback from

their peers. The group report was followed by a whole-class discussion that compared the features of the mathematical models produced by the various groups.

The problem provided rich opportunities for the group to engage in a range of mathematical practices, along with the development of important mathematical constructs of ranking, weighting ranks and selecting and aggregating ranked quantities that are embedded within the problem. The mathematical practices included interpreting and re-interpreting the problem information, making appropriate decisions, justifying one's reasoning, posing hypotheses and problems, presenting arguments and counter-arguments, applying previous learning, and acting metacognitively.

From English (2004)

In a three-year longitudinal study of elementary children's development of mathematical modelling, Watters, English, and Mahoney (2004) demonstrated how engagement in extended modelling problems provides opportunities for learners to engage in a range of mathematical processes and develop mathematical understanding. Because the modelling activities in the study were designed for small-group work, they also provided useful opportunities for developing collaborative problem-solving skills. In addition, children developed important metacognitive and critical thinking skills that enabled them to distinguish between personal and task knowledge and to know when and how to apply each during problem solution. Tanner and Jones (2002) worked with six secondary schools in Wales to investigate the use of mathematical modelling tasks with 24 classes of eleven- and twelve-year-old students. Using a matched quasi-experiment with pre- and post-tests and accompanying classroom observations and interviews, the researchers reported positive but small effect sizes when teaching interventions involved the explicit integration of metacognitive thinking skills such as planning, monitoring, and evaluating with the practical modelling activities.

As noted earlier, and in the Early Years chapter, learner-generated examples, interests, and questions provide a rich source of investigative-type activities for learners and, additionally, important feedback for teachers. The following vignette from Biddulph's (1996) research investigating the nature of questions that young learners ask about geometry illustrates how student questions can give insight into their thinking and provide a useful starting point for investigations.

Children's Questions

Biddulph (1996), in a small study involving 100 children aged eight to eleven years, noted that student-generated questions revealed considerable insight into their thinking and understanding. With respect to geometry, a proportion of the children's questions suggested that the children already had some understanding of particular concepts, for example:

- "How can you find the line of symmetry?"
- "Do you always need to draw squares of a grid to enlarge something?"

Other questions, however, revealed considerable lack of understanding, particularly with reference to angles:

- "Do angles cross?"
- "Are angles something that lean to one side?"

Biddulph noted that of the 73 questions in geometry, approximately 80% could provide the basis of worthwhile investigations. For example, the following questions about tessellations could be investigated together:

- "What shapes can be in a tessellation?"
- "Can you use two shapes to pave a driveway?"
- "Can it work with a shape with wavy lines?"
- "Does it change if the shape is different sizes?"

The children's questions in Biddulph's study also included some that illustrated that children's feelings are integral to their learning of mathematics. For instance, one child asked in exasperation, "I wouldn't have clue how to do it; what's the use of this?" Another child was concerned to know, "Is geometry safe?"

From Biddulph (1996)

Mathematics teaching for diverse learners ensures that tasks link to learners' prior knowledge and experiences

The learning task is its conceptual component; the learning activity is the task's practical counterpart, or the means through which the teacher intends the child to make the required conceptual advance from what was learned previously to what must be learned now. (Alexander, 2000, p. 351)

An important factor in the implementation of any task is the relationship between learner and task (Turner & Meyer, 2004). Tasks that provide students with opportunities for success, present an appropriate level of challenge and difficulty, and increase students' sense of control and arouse their interest can help elicit intrinsic motivation.

When students engage in tasks in which they are motivated intrinsically they tend to exhibit a number of pedagogically desirable behaviours including increased time on task, persistence in the face of failure, more elaborative processing, the monitoring of comprehension, and selection of more difficult tasks, greater creativity and risk taking, selection of deeper and more efficient performance and learning strategies, and choice of activity in the absence of extrinsic reward. (Middleton & Spanias, 1999, p. 66)

Conversely, if tasks are inappropriate in terms of motivation, interest value, or learners' prior knowledge—or simply because they lack suitably specific task expectations—student engagement may well be lower than anticipated or desired (Stein et al., 1996).

Planning task activities and learning goals

Planning appropriate learning sequences is an essential role for teachers (Cobb & McClain, 2001). Supported by research-based frameworks, there has been a move in recent times to design tasks around learning trajectories that signal important signposts for student learning (e.g., stages in the NDP Number Framework or 'growth points' in numeracy (Clarke, 2001)), and a substantiated means of supporting and organising this learning. Evaluation reports of the NDP and Te Poutama Tau consistently report that the Number Framework and associated Strategy Teaching Model have helped create for teachers a series of reference points for planning 'where to next?' tasks to meet individual needs (Higgins et al., 2004; Trinick, 2005).

Learning trajectories provide a general overview of the learning continua of the classroom community, not of individual students: students do not all progress along a common developmental path (van den Heuvel-Panhuizen, 2001; Wright, 1998). McChesney (2004), in a study of three New Zealand year 9 and 10 classes, found that student development of number sense was a nested and recursive process. Because students did not move through the stages of the Number Framework in a linear fashion, it was important that classroom activity "afforded opportunities for students to return to familiar mathematical entities, and negotiate further mathematical meanings" (p. 301).

When designing and implementing instructional tasks, it is important that learning goals and activities are adapted in response to teachers' perceptions of students' levels of understanding and in response to their ongoing evaluation of students' performance. Within the context of the objective-based numeracy lesson commonly found in English classrooms, Askew and Millett (in press) stress the need for flexibility that allows teachers to be "both responsive to students

and responsive to the discipline” (Ball, 1997). Their research found that in many instances the transformation of lesson objectives into meaningful learning experiences was mediated by teachers’ knowledge of the subject matter. To illustrate, Askew (2004b) provides an analysis of a lesson based on the stated objective, “to understand multiplication as repeated addition.” Based on observation, it was clear that the teacher’s intention was that children should use addition as a means of *calculating* multiplications. In the teacher’s account of the lesson, she emphasised that this meant that children have strategies that they understand. However, rather than building an understanding that multiplication is more efficient than addition, Askew concluded that some students could be building an understanding that addition is the foundation of multiplication. This could potentially lead to difficulties with understanding calculations such as $\frac{1}{2} \times 12$.⁶ The interpretation that ‘you can use addition to find answers to multiplication calculations’ was observed by Askew and Millet in several other numeracy classrooms.

Quality teaching based on appropriate sequencing also requires flexibility of task implementation. A concern raised by Askew and Millett (in press) relates to the flexibility with which teachers are able to work with students’ methods or the methods pre-specified in lesson objectives. Objectives that are too small (e.g., a learning intention ‘To add or subtract the nearest multiple of 10, then adjust’) may lose sight of the big picture. Askew and Millett’s account of a numeracy lesson exemplifies how a teacher, focused on helping children carry out a specific calculation strategy, can fail to help them make meaning from the tasks by not locating them within a broader network of mathematical ideas.

Managing learning trajectories through tasks that allow multiple entry points is the focus of an ongoing research project, Maximising Success in Mathematics for Disadvantaged Students (Mousley, Sullivan, & Zevenbergen, 2004). In their earlier project, Overcoming Structural Barriers to Mathematics Learning, these researchers found that the practice of having all students complete the same open-ended task, with the only difference being in expectations for performance levels, failed to recognise the needs of diverse learners. In an attempt to counter potential negative effects of differentiated learning *expectations*, the current three-year project examines an alternative practice of task differentiation in which the whole class completes the same basic task. Students experiencing difficulty are provided with alternative pathways, shaped by ‘prompt’ activities that provide stepping stones leading to the learning goal:

We suggest that a sense of community is more likely to result from teachers offering prompts to allow students experiencing difficulty to engage in experiences related to the initial goal task, rather than, for example, requiring such students to listen to additional explanations, or to assume that they will pursue goals substantially different from the rest of the class. (Sullivan et al., 2004, p. 259)

Within the study, students who finished a task quickly were posed supplementary tasks that extended their time on that task, rather than proceeding onto the next stage in the lesson. This approach was successfully trialled across several classes. Adjustment of the task, based on students’ responses, enabled all students to participate in class discussion and reviews and, most importantly, all students were prepared to move together on to the next stage of the learning. While the trial lessons found that this form of task differentiation changed the learning experience of students, especially for those who were at risk of not being able to follow learning trajectories set with the whole class in mind, the researchers noted the need to examine large-scale implementation of this approach, along with other forms of task differentiation.

Bicknell (2003) provides an example of an open-ended task suitable for whole-class work. Her account of a year 3–4 teaching unit on volume, based around the packing of boxes, shows how students were able to work at levels appropriate to their developmental stage. The research identified clearly distinguishable levels related to the children’s experience and proficiency with number strategies. Some children in the class moved easily from a model ‘of’ the packing situation towards a model ‘for’, generalising their strategies so that they applied to the volume of a rectangular prism.

Flexibility in task sequencing requires that teachers themselves engage deeply with the concepts involved. This engagement can occur both during planning and during teaching. In examining the planning process through a case study of one teacher, Mary, it emerged that lesson planning was as much a process of learning as it was a process of preparing for the contingent activity of teaching. Rosebery (2005) found that Mary conceptualised her own learning in much the same terms as those she used when thinking about her students' learning:

She expects to examine what she knows from a variety of perspectives, including past and present experience, 'official' meanings (e.g., from teachers' manuals, textbooks, [and mathematics advisor] and her students' understanding. ... Mary is not satisfied with her own understanding until she has integrated these potentially disparate points of view into a coherent whole. Nor is she ready to make decisions about teaching until she has achieved this. Thus for Mary, lesson planning is as much a process of learning as it is of teaching. (p. 323-4)

Teachers' report that supporting material in numeracy professional development projects (e.g., the New Zealand *Numeracy Development Project* and England's *National Numeracy Framework*) provides useful guides for knowing 'where to next', and being clear about 'what's expected' (Askew et al., in press; Higgins et al., 2005). Likewise, teacher development studies in the US report the benefit of increased teacher knowledge of students' thinking related to frameworks or progressions (e.g., Carpenter et al., 1997; Doerr & Lesh, 2003). In a professional development programme focused on typical milestones and trajectories of children's reasoning about space and geometry, Jacobson and Lehrer (2000) found that in those classes in which teachers were more knowledgeable, not only did students learn more than their counterparts, but this difference in learning was maintained over time.

Connecting tasks to learners' existing proficiencies and knowledge

Linking mathematics tasks to learners' existing proficiencies is a significant factor in the maintenance of high-level cognitive activity (Stein et al., 1996). Productive task engagement requires that tasks relate sufficiently closely to students' current knowledge and skills to be assimilated, yet be different enough to transform their methods of thinking and working (Grugnetti & Jaquet, 1996).

Fennema et al.'s (1996) longitudinal study of 18 teachers from the Cognitively Guided Instruction Project found achievement in concepts and problem solving was higher when instruction was designed around students' existing proficiencies and concept images. This approach replaced the more traditional approach where teachers focused on filling gaps in students' knowledge or remediating weaknesses. Likewise, increased performance gains recorded in the New Zealand NDP are, in part, attributed to teachers' increased awareness of their children's existing knowledge and the consequent focusing of instruction on "where students are at" (Higgins, 2003).

Understanding 'where learners are at' is not, however, always easy. In order to understand students' current thinking, teachers need to be able to deconstruct their own knowledge into a less polished, less final form—to work backwards from a mature and compressed understanding of the content in order to unpack its constituent elements (Ball, 2000). The following vignette illustrates how teachers and students can have different interpretations of the mathematics under discussion.

Dwindling Savings

Seventh grade students are discussing the expression $15000 - 300w$ for calculating somebody's dwindling savings as a function of the number of weeks (w) during which the money was spent.

Teacher: Would anyone do anything differently? Martha?

Martha: I'd do 15000 minus brackets, 30 and number of weeks [writes: $15000 - (300 w)$].

Teacher: All right. Do we need brackets around this? [points to 300 w]
Simon: Yes, you do, because you have to know that there's an operation. A person, now, he'll probably think *300 weeks*, not *300 times weeks*.
Teacher: OK, anyone who now knows algebra will know there is an operation.

From the exchange it appears that the teacher was unaware that the algebraic expression may have been initially interpreted by the students as an abbreviated sentence in which letters, such as w, were a shorthand for nouns, such as weeks, rather than placeholders for numbers.

From Sfard (2005)⁷

The episode also illustrates that, in any learning situation, children create their own meanings—meanings that sometimes form useful building blocks and at other times are not appropriate at all. Here ‘appropriateness’ refers not to the inner coherence of students’ knowledge but to possible disparities between students’ conceptions and the public versions of the same idea (Davis, 1998). For example, many young children quite sensibly transfer the belief that multiplication makes bigger and division smaller to fractions, at least in the first instance. As such, it is important to regard the appearance of misconceptions as a natural part of the learning process rather than a contradiction of a vision of learners as sense-makers:

Indeed, not in spite of students’ need for meaning but rather because of it, students tend to construct their own conceptions. Precisely because of their need to fit new concepts into their former knowledge, their understandings are sometimes at odds with the official definitions. (Sfard, 2003, p. 357)

Teachers who understand their students’ thinking as ‘understanding in progress’ are typically able to use such misconceptions as building blocks (see Groves and Doig, 2004), creating tasks that challenge misconceptions and provide opportunities to resolve cognitive conflict.

The presentation of a possible case for conflict is much more than an attempt to enculturate the student into conventional understandings, or to extend the student’s experience of a particular discourse, it is an induction into an internal validity of mathematics by offering the student an opportunity to rethink and restructure existing assumptions and understanding, or at the very least to realise that there is a problem. (Watson, 2003, p. 31)

Watson illustrates this with an episode involving a student who thinks that the derivative of x^2 is $2x + k$. To continue the negotiation, the teacher asks for the derivative of $x^2 + kx$. By offering a very particular example, the teacher informs the negotiation process—not by judging the student’s ability but by giving the student the responsibility for sorting it out for herself.

In order to link tasks to students’ thinking and experiences and to make appropriate choices regarding the difficulty level and degree of task explicitness, teachers need detailed knowledge of their students. While such knowledge can be built up from a range of assessment practices, listening to students’ mathematical explanations and justifications is seen as a necessary component (Davis, 1997). Teachers, in listening, need to notice significant mathematical moments and respond appropriately. This is a complex task (Sherin, 2001). Watson (2000), in a UK study of 30 primary, middle, and secondary teachers’ informal assessment practices, expressed concern about the value of the ‘explain how you did this’ approach. Students’ oral contributions more often reported mathematics already ‘done’, thus limiting teacher access to how their students are thinking mathematically. A recent New Zealand study (Davies & Walker, 2005) reports on the use of classroom videotapes to focus teachers’ attention on the mathematics in classroom interactions. Teachers reported that increased awareness of significant mathematical moments meant that they were more likely to respond to the mathematical learning needs of their students. As one teacher pointed out:

Something that I’ve learned this year is that you can let the children guide a lot of the learning. It’s OK to stop and smell the roses a little bit. If something comes up, that’s a great little teaching moment—go for it—grab it—even if it goes off

on a tangent somewhere, rather than having to just stick to the book—close the book—someone’s brought up that idea so let’s talk about that. (p. 278)

Frameworks, such as the Number Framework in the NDP (Ministry of Education, 2006), provide a basis for understanding critical stages in children’s development. When used in conjunction with a diagnostic interview focused on students’ understanding and solution processes, frameworks can usefully map children’s thinking (Clarke 2001; Irwin & Niederer, 2002; Thomas & Tagg, 2004).

The assessment interview has given focus to my teaching. Constantly at the back of my mind I have the growth points there and I have a clear idea of where I’m heading and can match activities to the needs of the children. But I also try to make it challenging enough to make them stretch. (Clarke, 2001, p. 20)

Teachers involved in numeracy professional development (e.g., in New Zealand, Australia, and England) reported being surprised (at least in the initial interviews) at what many children were or were not able to do:

In every class there is that quiet child you feel that you never really ‘know’—the one that some days you’re never really sure that you have spoken to. To interact one-to-one and really ‘talk’ to them showed great insight into what kind of child they are and how they think. (Clarke, 2001, p. 14)

Teachers in all of these numeracy projects report that their use of diagnostic interviewing has helped them move to a more responsive pedagogical practice that uses students’ existing mathematical knowledge as a basis for task design and instructional direction.

Situating tasks in contexts

Across all curricula, opportunities to explore authentic applications that arise out of real-life contexts can have a significant and sustained impact on student knowledge, attitude, self-esteem, independence, and confidence (Alton-Lee, 2003). In order to make mathematics more meaningful and accessible for all learners, mathematics curricula frequently advocate the use of contexts. In this sense, ‘context’ refers to a real or imaginary setting for a mathematical problem, which illustrates the way the mathematics is used.

Advocates claim that the use of contexts can motivate, illustrate potential applications, provide a source of opportunities for mathematical reasoning and thinking, and anchor student understanding (Meyer, Dekker, Querelle, & Reys, 2001). For example, Wiest’s (2001) study of 273 year 4 and 6 children’s problem solving found that the selected contexts affected a range of variables, including the children’s interest in, attentiveness to, and willingness to engage with problems, the strategies they used, their effort, their perception of and actual success, and the extent to which they learned the intended mathematics. For these young children, fantasy contexts evoked stronger responses than other contexts. Wiest argues that fantasy contexts offer an appealing link to abstract and creative thinking but notes that, as with any ‘contextual’ motivator, it is more important to seek long-term benefits than short-term ‘feel-good factors’. Moving from fantasy to reality, several research studies (e.g., Watson, 2000) have found newspaper reports to be a useful source of contextual problems. Arnold (2005) provides examples of calculus problems related to ozone depletion and IBM share prices that he has used with senior students, claiming that there is much value in asking students to think critically about what they read in the news.

Internationally, the most prominent advocate for contextually-based tasks is the Dutch Realistic Mathematics Education (RME) movement. In this programme, Gravemeijer (1997a) maintains that the primary use of context is not to motivate students but to provide a learning situation that is experientially real for the students⁸ and which can be used as a starting point for advancing understanding. “What is important is that the task context is suitable for mathematisation—the students are able to imagine the situation or event so that they can make use of their own experiences and knowledge” (van den Heuvel-Panhuizen, 2005, p. 3) to support the development

of context-related strategies and notations that later become more generalised. For example, when solving problems such as dividing three pizzas among four children, students arrived at solution methods that pre-empt the more formal procedures for adding and subtracting fractions with unequal denominators (Streefland, 1991).

Watson (2004) also argues that ‘realistic’ does not mean that tasks must *necessarily* involve real contexts. Citing a student of Goldenberg (1996), she advocates that tasks should be seen as ‘realistic’ not because they relate to any particular everyday context, but because they make students think in ‘real’ ways. Watson noted that students in the Improving Attainment in Mathematics Project were usefully motivated and intrigued by tasks that exemplified the ‘power’ of mathematics. Similarly, Ainley and colleagues (2005) claim that the purposeful nature of activity is a key feature of relevance:

This use of ‘purpose’ is quite specifically related to the perceptions of the learner. It may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a ‘real world’ context. (Ainley, Bills, & Wilson, 2005, p. 18)

Purposeful tasks provide students with opportunities to use and learn about particular mathematical ideas in ways that allow them to appreciate their utility. Ainley et al. (2005) provide an example of students solving a ‘pyramid’ task in the form of a contextualised game. Students who had an appreciation of the purpose of the exploration, which was to find *how* to get the highest total (as opposed to *calculate* the highest total) were more likely to attend to the underlying mathematical structure of the array output in the spreadsheet. In addition, some students were able to appreciate and articulate the utility of standard notation (in this case, Excel formulas) for clarifying the ways in which the total was calculated.

Research findings (e.g., Burton, 1996; Salomon and Perkins, 1998) demonstrate that the use of context in mathematics tasks does not automatically lead to improved learning outcomes for all students. Sullivan et al. (2002) argue that problem contextualisation may actually contribute to the disempowerment of some students. Certain contexts that are taken as realistic and relevant may in fact restrict the mathematical development of students from socially and culturally divergent backgrounds.

Lubienski (2000) found that contextualising tasks obscured their mathematical purpose for some students. In particular, students categorised as lower SES tended to focus on the contextual issues of a problem at the expense of the mathematical focus and treated problems individually without seeing the mathematical ideas connecting them. Likewise, Cooper and Dunne’s (2000) review of the national testing system in the UK found that contexts created another layer of difficulty for some students—particularly those from working class backgrounds who appeared to approach the problems in heavily context-laden ways unintended by test writers. In New Zealand, Forbes (2000) raised similar concerns. In her study on assessment, Māori students appeared to be disadvantaged by the contextual information in the tasks. The tangible, ‘concrete’ context expected to assist students was not necessarily concrete in the sense of making sense. Meaney and Irwin (2005), in their investigation of four tasks from the National Education Monitoring Project (NEMP), found that year 8 students were far more successful at recognising the need to ‘peel away’ the story shell of the problems. A year 4 response to the request to work out how many cars went down a motorway in 9 minutes if 98 cars went down every minute was: “I think 259 ... because, umm, there’s lots of car going up and down, and um, cars like going visiting and on the bridges.” In another investigation involving NEMP tasks, Anthony and Walshaw (2002) noted a similar trend, with year 4 students more frequently accessing their informal knowledge. In their attempts to solve problems, students’ real-world concerns sometimes collided with the mathematical solution: When asked to describe “How much of the pizza is left?” a year 4 student responded, “All the herbs.”

Rather than abandon the use of contextual tasks, Sullivan, Zevenbergen, and Mousley (2002) argue for more critical task selection and implementation. Tasks that simplify real situations

unrealistically (see discussion on word problems in van den Heuvel-Panhuizen, 2005) or use mathematics to solve problems in unrealistic ways (e.g., working out the number of slices in a cake from the number of decorative patterns on a slice) or use situations which would be unfamiliar to all students (see Taylor & Biddulph, 1994, for context issues in probability) should be avoided.

Deciding which contexts are familiar to students is challenging. What might appear on the surface to be suitable context is not necessarily one with which all children are familiar, as is evidenced in a study of seven- and eight-year-old New Zealand children's experiences with money (Peters, 1995). Based on interviews and role play, Peters concluded that money is not a familiar context for addition and subtraction. All the children who were able to spend money accurately did so one coin at a time or purchased one item at a time. Given that this research was completed some 10 years ago, before EFTPOS transactions were introduced, it seems even more likely that programmes that use money as a context for learning early arithmetic skills will not necessarily reflect young children's ideas or experiences about money. Brenner (1998) and Guberman's (2004) investigations of children's experiences of learning about money at home and at school (see also discussion on money activities in chapter 6) also note that differences between everyday and school mathematics can make the inclusion of money contexts problematic, especially in relation to decimal notation. Rather than discard money contexts, Brenner suggests teachers infuse the curriculum with more open-ended problems and provide opportunities for student–teacher discourse to help bridge the gap between everyday and school mathematics. In connecting everyday commonsense with the curriculum, it is essential to acknowledge that “the connection does not lie in the curricular materials but in the discussion and reasoning that can take place when such materials are used” (p. 153).

While not all mathematical tasks need originate from students' cultural experience, it is necessary that embedded contexts are accessible to all students. Embracing culturally contextualised pedagogy is not, however, simply a matter of incorporating ethnic symbols and artefacts into tasks. McKinley, Stewart, and Richards (2004) report that the superficial use of a small number of “Māori contexts” in junior secondary mathematics school programmes can lead to these cultural activities being seen as “Māori caricatures”. In an effort to provide a more sound pedagogical approach, Anderson and her colleagues (2005) describe the successful implementation of tukutuku panel activities into their pre-service mathematics programme. Programme evaluation reported support for the inclusion of cultural activities; students felt confident that they could integrate these within future mathematics programmes.

Clarke (2005) advocates the role of context as a social construction:

Context in our view is neither a neutral background for the negotiation of mathematical meanings, nor merely a catalyst mediating between tasks context and the individual's mathematical tool kit. Rather we should speak of the personal task context as an outcome of the realization of the figurative context within the broader social context. (Clarke & Helme, 1998, p. 130, cited in Clarke, 2005)

With reference to recent curriculum examples from South Africa, Clarke illustrates how the incorporation of themes of societal significance can affect students' engagement with, and perceptions of, mathematics. Analysis of student–student interactions in South African classes found that AIDS provided a genuine context that allowed the nature and purpose of the classroom activity to be constructed in socio-cultural—not just mathematical—terms. Clarke argues that dichotomisation of real-world and school mathematics was avoided by “viewing students as simultaneously members of complementary communities of practice within a broader integrative socio-cultural context” (p. 9).

Sullivan, Mousley, and Zevenbergen's research project, *Overcoming Structural Barriers to Mathematics Learning*, aimed to address those factors inherent in the culture of schooling that constrain some students (often those from low-SES backgrounds) from engaging in context-based tasks. Multiple case studies in primary school classrooms have examined pedagogical

strategies in which teachers make *explicit* those hidden aspects of pedagogy that can inhibit student participation in open-ended contextually-based tasks. In Sullivan et al.'s 2003 study of three grade 6 teachers teaching a graphing activity based around a worm farm context, the explicit pedagogical factors were in all three cases related to student activity and outcomes. All three teachers made it explicit that there could be multiple responses to the task, but only two emphasised to their students that they could be creative in their responses. It was observed in one class that a lack of direct reference to the practical and creative possibilities of the task, combined with an explicit emphasis on the mathematical aspect, constrained the outcomes. As one observer noted:

On looking at the work, they seemed to have done mathematically correct descriptions of the graph, but they were not particularly creative in their interpretations. The issue of their initial lack of creativity was interesting. Perhaps they have too little scope in their knowledge of worm farms to be creative. Alternatively perhaps they did not link the creative interpretation with maths. (Sullivan et al., 2003, p. 271)

The use of everyday contexts acknowledges the fact that students of all backgrounds have commonsense understandings of mathematics “that need to be harnessed and reconciled with the more general and powerful mathematical knowledge that students learn in school” (Brenner, 1998, p. 153). Learners’ real-life, circumstantial knowledge is referred to in the literature as ‘intuitive knowledge’ (Resnick & Singer, 1993), ‘situated knowledge’ (Brown, Collins, & Duguid, 1989) and ‘informal knowledge’ (Saxe, 1988). Several New Zealand studies (e.g., Hunter, 2002; Irwin, 2001) claim benefits for the use of everyday contexts—both to support mathematical learning from informal and prior knowledge and to challenge misconceptions. Hunter and Anthony (2003) outline a teaching approach to decimals that involves using percentages and filling containers with water. The activity provides a visible representation of decimals that connects with students’ everyday experience.

Irwin’s (2001) research investigated 11- and 12-year-old students’ learning of decimals. Pre- and post-testing revealed that students who worked on contextual problems made significantly more progress in their knowledge of decimals than those who worked on non-contextual problems. Analysis of paired interactions suggested that greater reciprocity existed in the pairs working on the contextualised problems. Irwin suggested that this was partly because, for contextualised problems, students who traditionally were lower achievers made greater use of their everyday knowledge of decimals. In addition, Irwin claimed that requiring students to link their scientific and everyday knowledge gave them opportunities to confront a range of misconceptions such as ‘one-hundredth’ is the same as 0.100 or ‘ $\frac{1}{4}$ can be written as either 0.4 or 0.25’. Like other researchers, Irwin found that the benefits of contextualising are mediated by the match that contexts make with students’ experience and misconceptions. As such, problems involving exchange rates (seen by students as relevant because of their parents’ experience of overseas travel and the practice of sending money overseas) were selected ahead of the more typical textbook contexts involving cricket or baseball statistics. The teachers in Irwin’s study noted that information on relevant contexts was readily obtained through discussion with students.

In a three-year study involving year 5 learners of English as an Additional Language (EAL), Barwell (2005) also found there were benefits to be gained by linking word problems to students’ informal knowledge. Earlier research suggested that the use of word problems could prove a barrier to mathematics students with low levels of proficiency in both their native language and English (Clarkson, 1992), but Barwell found that the use of a familiar context facilitated a shared sense for the emerging word problem and its solution as well as facilitating students’ social relationships.

Pocket Money

In a study spanning 3 years, Barwell (2005) explored the nature of participation of year 5 learners of English as an additional language in mathematics classroom interaction. His particular focus was on how these students made sense of arithmetic word problems. Barwell found that when mathematics is the focus students do not bring to the foreground their difficulty with the English language or their use of it. Instead they use their personal experiences of the issue under consideration in the word problems as a starting point for student discussion.

Word problems included situations around furniture stores, pocket money, McDonalds, and pizzas. The students all used their individual experiences as a general backdrop for discussions about the details of the word problem. For example, in a problem involving pocket money, the students' discussion included consideration of whether £5 is a sensible amount and how much pocket money they each received, in order to develop a shared understanding of the context of the problem. Those personal experiences and their accounts of them, operated to develop a shared sense of context between mathematical partners and helped to establish a common goal to find a mathematical solution. Those common experiences also served the purpose of creating a sensibility towards a solution's meaningfulness and its relevance. "The study highlights the close interweaving of the negotiation of participant's relationships and their work on the word problem task" (p. 345).

The provision of a shared learning experience may go some way to counter the detrimental effects created when knowledge of a problem context is not distributed evenly among students from different social backgrounds. Barwell suggests that links to personal experience may be significant for all learners but particularly significant for learners of EAL, who may not so readily have access to the discourse of school mathematics.

From Barwell (2005)

In addition to issues of relevance, the language demands of contextual problems can be a significant barrier for some students (Draper & Siebert, 2004). Recent studies (Koedinger & Nathan, 2004; Lawrence & Patterson, 2005; Zevenbergen, Hyde, & Power, 2001) have found that many senior secondary students have ineffective strategies for dealing with word problems. Many students base their interpretations on trigger words such as 'more' (suggesting an additive operation). Anthony (1996b), in a year-long study of a year 12 class, reported the frequent teacher use of keywords as an instructional strategy. For example, in a review session, the teacher made these comments:

- T: When you hear 'gradient' or 'tangent', what should you think about?
- T: What are the keywords—what should the words 'rate of change' tell you?
- T: Look at the paper—the most important thing is to find the keywords.

While a keyword may help some students *complete* a problem, keywords do little to help students construct meaningful mathematical knowledge. By compensating for skill and knowledge deficits, keywords enable students to complete problems without necessarily understanding the situation, without modelling it mathematically, and without acquiring the intended procedural knowledge.

Situational contexts can give students access to mathematics, but research has shown that they should not be the goal of a learning experience nor should they be allowed to blur the focus of the mathematics.

Providing appropriate challenge

When selecting a task, all teachers consider the level of challenge it presents. Teachers who provide moderate challenges for their students signal high expectations, and their students report higher self-regulation and self-efficacy together with a greater inclination to seek help (Alton-Lee, 2003; Middleton, 2001). Mathematical tasks that are problematic and offer

an appropriate degree of challenge have high cognitive value. Tasks that are too easy or too hard have limited cognitive value (Henningsen & Stein, 1997; Williams, 2002). Francisco and Maher (2005) report on a longitudinal study involving a group of students, tracked from grade 1 to university, engaged in well-defined, open-ended mathematical investigations. The study involved an examination of (a) videotapes of the students' mathematical behaviour and written work in problem-solving activities, (b) student reflections on their experiences, which were collected during clinical interviews in grades 11 and 12 and via follow-up questionnaires, and (c) results of detailed analyses of the development of students' ideas and ways of reasoning. The researchers claimed that providing students with the opportunity to work on a complex task—as opposed to a series of simple tasks devolved from a complex task—was crucial for stimulating their mathematical reasoning and building durable mathematical knowledge: “The opportunity to attend to the intricacies of a complex task provides the students with the opportunity to work on unveiling complex mathematical relationships, which enhances deep mathematical understanding” (p. 371).

Research has consistently documented differences in levels of challenge in the tasks provided to students of different ability groupings. The QUASAR project (Stein et al., 1996) was based on the premise that prior failures of poor and minority students in the US were due to a lack of opportunity to participate in meaningful and challenging learning experiences rather than a lack of ability or potential. Houssart (2001) found that teachers in England tended to offer challenging tasks to ‘higher’ sets and use step-by-step approaches with ‘lower’ sets. Teachers of ‘higher’ sets showed more enthusiasm for investigative tasks that encouraged creativity:

There has to be an element of challenge about it ... they want to be tested in what they're doing and not feel they're doing something babyish or below them.

Challenge, especially with the top set ... Probably, had I had the lower set, the challenge bit would ... be far lower down, until they got the basics in obviously.

When describing a ‘good’ task, many of the ‘bottom set’ teachers in Houssart’s study talk in a way that contrasts starkly with ‘top set’ teachers’ talk of challenge and creativity:

They like colouring in. I always start with the concrete and what they know first, we did some little games ... games are successful.

Of concern is Houssart’s (2005) finding—noted as striking—that many of the low-attaining students in her study preferred more challenging tasks.

Sometimes it was just a case of the task getting harder, at other times it was making it less routine or repetitive. ... The most striking finding was that whenever a child refused to do a mathematical task, there was a possible explanation in the nature of the task, even if other factors contributed. Even those children coming to mathematics lessons, apparently unprepared to do anything eventually opted in as different tasks were offered, and there was always a possible explanation in the nature of the task itself. (p. 72)

From the Australian literature, we find similar reports. Senior teachers from two secondary schools reported the use of explanatory or investigative methods with ‘able’ students and ‘show and tell’ with ‘less able’ students (Norton, McRobbie, & Cooper, 2002). Only a small number of the 162 primary school teachers surveyed by Anderson (2003) indicated that *all* students could learn by doing open-ended and unfamiliar problems on a regular basis. Anderson offers this response from an experienced teacher as illustrative of the reasons why this might be so:

It's safer—children feel more comfortable if they're not made to think. I realise this is cynical—but for many children with low IQs and poor/non-existent English language skills, the concept of problem solving is alien. Also it takes up too much time and there is great pressure to “get through” the curricula. So whilst in theory I acknowledge the potential of problem solving, in reality with some clientele it's too hard. (p. 76)

Zevenbergen (2005) examined students’ experiences in streamed classes in secondary schools.

Based on interviews with 96 students and 10 teachers from six quite different schools, she reported that differences in pedagogical practices related to access to mathematical opportunities as evidenced by content coverage. After completing a unit of work, the students sat a common test consisting of three levels of questions that varied in complexity and application. Students from high-streamed classes reported that their lessons covered more content than required by the test. In contrast, lower-streamed students reported:

- Jaclyn: When it comes to exams, we can't do the work and can only get low marks. Which means we have to stay in the dumb class. I get so annoyed and just want to leave it but you know, you can't do anything about it. (p. 614)
- Becky: I would like to be in the top classes [because] they get the good teachers and they can learn the stuff and then do well in the exams. We are lucky if we can pass. We're not idiots, but the teachers think we are. (p. 616)

These studies and others challenge pedagogical practices based on simplification and repetition for low-achieving students. From her research with low-attaining students, Watson (2002) proposed an alternative approach focused on mathematical thinking. Using tasks selected by the classroom teacher, Watson taught the class, integrating a series of prompts (see Watson & Mason, 1998) designed to encourage pattern generalisation and the use and generation of examples, communicate a sense of mathematical concept, and describe underlying structures. The low-attaining students in her study showed that they were able to develop the capacity for exemplifying, generalising, abstracting, reflecting, and working with structure and images—kinds of thinking that marked them as successful novice mathematicians.

Following on from this study, [Watson and De Geest \(2005\)](#) worked with teachers in the Improving Attainment in Mathematics Project (IAMP) to enhance instructional practices that would support the mathematical thinking of students previously identified as low attainers. Based on their belief that these students were entitled to access mathematics, they chose not to simplify mathematical activities. They planned tasks that encouraged links with previous learning and were responsive to students' responses. Rather than use worksheets, the teachers were likely to develop tasks from starter tasks and the students' own questions. All teachers used 'create your own example' tasks as part of their everyday lesson structure and several used 'if this is the answer, what is the question?' tasks. In class, students were given more thinking time to complete tasks.

Matching tasks to the unique learning characteristics of students is particularly important for learners with special needs. Working with second and third grade students identified as learning disabled, Behrend (2003) found that, given the opportunity, they were capable of sharing their computation strategies, listening flexibly to other children's strategies, discussing similarities and difference in strategies, justifying their thinking, and helping each other understand word problems. In addition, the students were capable of generating and utilising their own problem-solving strategies and did not need to be taught specific strategies. Similarly, Thornton, Langrall, and Jones (1997) detail classroom episodes in which elementary students with significant learning disabilities successfully engaged with rich and meaningful problem tasks. For example, students tackling the problem, *Is every triangle $\frac{1}{2}$ of a rectangle? Yes or no? Prove it*, demonstrated multiple solution strategies based on the cutting and re-forming of triangles and extended the task to conjecture that every triangle is half a parallelogram. The conjecture raised questions about the defining properties of shapes—for example, when is a parallelogram a rectangle?—providing an opportunity for the teacher to follow up on the distinction between congruent shapes and shapes that have the same area.

Quality teaching provides intellectually and academically gifted students with appropriate task opportunities. Bicknell and Riley (2005) report, however, that a recent national survey found that teachers identified proportionally fewer Māori students as gifted and talented, and that this group of students is under-served in terms of culturally appropriate programmes. Without appropriate challenge, gifted students are 'at risk'; they may demonstrate boredom,

loss of interest in or commitment to mathematics, limited metacognition, and poor behaviour (Diezmann, & Watters, 1997). In accord with the policy of curriculum differentiation advocated for gifted students, (van Tassel-Baska, 1997), Diezmann and Watters (2004) suggest increasing challenge by task problematisation. Without changing the mathematical focus, a task can be problematised by methods such as inserting obstacles to the solutions, removing some information, or requiring students to use particular representations or develop generalisations. For example, the task of summing the numbers from 1 to 10 could be problematised by asking for a generalisation for the sum of the numbers from 1 to 100.

Diezmann and Watters (2002), reporting on a study involving twenty 11- to 12-year-old mathematically gifted students drawn from four mixed-ability classes, found that problematised tasks, when combined with a responsive teaching/learning environment, provided opportunities for gifted students to engage in productive mathematical activity requiring higher level cognition. “[S]tudents displayed greater persistence, collaborated with peers, demonstrated flexibility in thinking, checked work, and questioned each other” (p. 82). Studies involving younger children found that successful adaptations of tasks also included modification of games and extending manipulative use. For example, the construction of a number line to represent the distance of the ten brightest stars required the application of knowledge of large numbers, relative magnitude, and scale—and extension of the more typical use of the number line to represent sets of numbers (Diezmann & English, 2001a). According to Diezmann and Watters, the advantage of using these lateral strategies, as opposed to add-on or extension tasks, is that problematising, adapting, and enriching regular curriculum tasks provides underachieving gifted students with the opportunity to oscillate between regular activities and more challenging activities according to their capability, confidence, and motivation.

The provision of mathematical challenge is integrally linked to productive learning communities—the level of challenge affects students’ involvement in the task, both in terms of opportunities for mathematical reasoning and in levels of engagement. In a comparative study of two teachers, Groves and Doig (2004) explored features of the teachers’ actions. The first taught a year 1 class in Japan and the students were working on addition. The other taught an Australian year 7 class, where the students were investigating the area of a triangle. Grove and Doig explain how both teachers were able to transform familiar mathematics into challenging and enriching experiences for their students. In the teachers’ views, mathematical lessons were akin to dramas that have a climactic ending. To enact the dramatic interplay, the teachers asked thought-provoking questions, they lifted the level of mathematical discussion through their discursive skills, and they pressed for understanding.

Mathematical tasks need to provide opportunities for cognitive engagement and press for understanding

The basic aim of a mathematics lesson is for learners to learn something about a particular topic. To do this, they engage in a task ... the purpose of a task is to initiate activity by learners. (Mason & Johnston-Wilder, 2004)

“To facilitate long-term learning students need curriculum-appropriate opportunities to *develop new understandings* and to *practise and apply* their new learning” (Alton-Lee, 2003, p. 61). The development of mathematical understanding requires that learners have the opportunity and space to do ‘appropriate things’. These ‘things’ have variously been referred to as mathematical practices (Rand Mathematics Study Panel, 2003), mathematical processes (Ministry of Education, 1992), and mathematical thinking and reasoning (Fraivillig, Murphy, & Fuson, 1999). They include: “sort, classify, structure, abstract, generalise, specialise, represent and interpret symbolically and graphically, justify and prove, encode and decode, formulate, communicate, compare, relate, recognise familiar structures, apply and evaluate applications, and automatise” (Watson, 2004, p. 364).

Classroom tasks and activities provide the vehicle for students’ cognitive engagement, the

quality and level of which has a profound effect on mathematics learning outcomes (Helme & Clarke, 2001). Student cognitive engagement, qualitatively different from time on task or student participation, is influenced by the nature and implementation of available tasks. Research conducted in the QUASAR project indicates that when teachers choose tasks that require a high level of cognitive demand and set them up and implement them in ways that maintain a high level of cognitive demand, the result is an increase in student understanding and reasoning (Stein & Lane, 1996).

Stein and colleagues (1996) explored how different task demands, high or low, are placed on students. Lower level approaches to a task include memorising or reproducing learned facts, rules, formulae, and definitions, and using standard procedures or algorithms. Tasks that present high-level demands also use procedures, but in ways that build connections to mathematical meaning or require complex and nonalgorithmic thinking. Factors that support engagement with mathematical practices in the face of complex task demands include appropriate scaffolding (Anghileri, 2002), the modelling of high-level performance by the teacher and/or capable peers, the making of conceptual connections (Kazemi & Franke, 2004), the provision of appropriate amounts of time for exploring ideas and making connections (Stein et al., 1996), the encouragement of student self-monitoring (Pape et al., 2003), a sustained press for explanation, meaning, and understandings (Fraivillig et al., 1999), and the selection of tasks that build on students' prior knowledge. Many of these factors have been discussed earlier in this and the preceding chapter, so this section presents research findings specifically related to task implementation.

When implementing tasks in the classroom, it is simplistic to consider tasks as 'set' or 'fixed' (Henningsen & Stein, 1997). Each task has a relative cognitive value for an individual, and one cannot assume that the students' interpretations of that task—the activities that they engage in—are either similar to each other's or fit with the expectations of the teacher (Askew, 2004a). Stein et al.'s (1996) study of task implementation within secondary schools found that the higher the demands that a task placed on students in the set-up phase, the less likely it was that the task would be carried out faithfully during the implementation phase.

Over half (53%) of the tasks that were set up to require the use of procedures with meaningful connections failed to keep the connection to meaning alive during implementation. ... it appears as though follow-through during the implementation phase is most difficult for those kinds of task that reformers ... have identified as essential to building students' capacities to engage in the processes of mathematical thinking. (p. 476)

The factors that contributed most frequently to the lowering of task demands were (a) the challenging aspects of the task, (b) a shift in focus from understanding to correctness or completeness, and (c) inappropriate allocation of time. Other factors included the teacher relaxing accountability requirements and lack of alignment between the task and students' prior knowledge, interest, and motivation.

There are similar reports of lowered cognitive demands in UK classrooms. Watson (2002) reports that teaching mathematics to low-attaining students in secondary school "often involves simplification of the mathematics until it becomes a sequence of small smooth steps which can be easily traversed" (p. 462). Frequently the teacher will take the student through the chain of reasoning and the learner merely fills in the gaps with the arithmetical answer, or low-level recall of facts. This 'path smoothing' is unlikely to lead to sustained learning since the strategy deliberately reduces a problem to what the learner can already do—with minimal opportunity for cognitive processing. This pattern of participation further reinforces the view that if students sit and do nothing for long enough, the teacher will change the requirements so that the task can be completed with minimal effort.

New Zealand classroom research (e.g., Anthony, 1994; Walls, 2004) also documents task episodes that start out as cognitively demanding and, during implementation, become rather less demanding. Based on observations of a year 12 class, Anthony (1996a) noted that reduction

in task complexity occurred when students pressured the teacher to provide explicit procedures for completing the task or when the teacher ‘took over’ difficult parts of the task on the students’ behalf. Cognitive load was reduced by subdividing tasks, by setting short-term learning goals, or by providing informational products. For example, teacher-supplied helps such as a list of the items in a test, keywords, tables and topic summaries, all intended to support student learning, in reality substituted for student learning. This whole-class scaffolding on the pretext that all students will benefit creates a paradoxical situation. If the teacher provides unnecessary support for students who have the ability to accomplish a task without support, the relative cognitive value of a task is reduced. Unnecessary scaffolding can inhibit rather than facilitate students’ learning because “some students are able to circumvent task demands or work at tasks that are below their level of ability” (Doyle, 1983, p. 180).

In addition, Anthony (1994) noted students’ repeated efforts to resist task engagement. Practices regularly employed by students to reduce the cognitive load associated with a task included copying work from others, answering questions using prompts from other students, using the answers found in the back of the textbook, or offering provisional answers (guessing) to indicate apparent engagement in the task.

Stein et al. (1996) found that a classroom focus on the completeness or accuracy of answers was another significant factor associated with decline in task demand. In a series of studies involving elementary classes, Turner and Meyer (2004) also found that the “positive aspects of challenging students’ thinking often are circumvented by the reliance of students and teachers on superficial indicators of understanding, such as work completion, quick responses, or correct answers” (p. 311). Students and teachers frequently seemed willing to trade the benefits of challenge-seeking (competence, pride, efficacy, and enjoyment) for the safety of avoiding mistakes and appearing competent.

In New Zealand, Walls’ (2002) longitudinal study of ten primary school students noted teachers’ frequent reference to children’s ability in terms of completion rates rather than mathematical understanding.

I’ve inherited some problems I think from other years. Standards haven’t been set and kids just don’t complete work and they’re not use to getting, not used to actually getting through something. Finish it off. That’s something I’m very tough on. I like things to be completed. (Early year 3) (Walls, 2002, p. 205)

According to Walls, this orientation towards completion appeared to be reinforced by teachers’ interactions with children:

Ms Summers: (To Peter) You’ve finished! Doesn’t it feel good when you’ve done it? (Late in Y 3)

Mrs Kyle: How many finished? (Looking around at the show of hands) Most of you didn’t finish. You must learn to put ‘DNF’—did not finish, at the bottom. (Early in Y 4)

Ms Torrance: We have some amazing speedsters who have got on their rollerblades and got their two sheets done already. (p. 206)

Collectively, these studies suggest that a classroom orientation that consistently defines task outcomes in terms of the answers rather than the thinking processes entailed in reaching the answers negatively affects the thinking processes and mathematical identities of learners.

In contrast, advocates of a problem-solving curriculum (e.g., Stacey, 2005 and also earlier discussion in this chapter) argue that teaching *through* problem solving supports high-level cognitive activity.

Problem solving activities are prominent in the recommended task only to the extent that they support the learning of other parts of the curriculum. In this vision, open problem solving is absorbed into normal teaching, as an attitude to learning and a process underpinning achievement in the normal curriculum. The goal of the

tasks highlighting problem solving is to promote good learning of routine content and to develop useful strategic and metacognitive skills, rather than explicitly to strengthen students' ability to tackle unfamiliar problems. (Stacey, 2005, p. 349)

Complex Numbers in Carmel's class

The teacher, Carmel Schettino, reports the following 'success story' from her attempts to implement a problem-solving curriculum in her pre-calculus (upper secondary school) class.

I assigned a problem that required the students to simplify the expression $(4 + 3i)/(1 + i)$. We had discussed using the conjugate of a complex number to rewrite reciprocals of complex numbers in $a + bi$ form, but I had not given an example using conjugate multiplication as the method of 'dividing' complex numbers. However, we had discussed the transformational ramifications of the multiplication of complex numbers being a rotation and dilation of the complex number. Stephanie and Kara went to the board with their solution, although they were sure that their method was incorrect. They had collaboratively extended those transformational ideas to division, assuming that the simplified complex number resulted from a clockwise rotation (subtracting the angles) and reduction of the radius (the new radius was the quotient of the two old ones). The rest of the class was impressed with the students' ingenuity and the use of the tools to which they had already been exposed. Catie then suggested using conjugate multiplication, and Stephanie and Kara were surprised that the two answers were the same in rectangular form.

According to the teacher, this episode illustrated the creativity that was a more frequent outcome of her new problem-solving curriculum. She noted that Stephanie and Kara had come a long way from the beginning of the year. Instead of the common retort, "I have no clue," they were now able to develop an innovative method for solving the problem.

From Schettino (2003)

Stein et al.'s, (1996) study of students' engagement in mathematical tasks identifies those factors that assist the maintenance of high-level cognitive activity. The most frequent factor (in 82% of the tasks that remained high-level) was that the task built on students' prior knowledge. The second-rated factor (71%) was the provision of an appropriate amount of time for task execution. Evidence of teacher scaffolding of mathematical activity was found in 58% of the tasks that remained at high cognitive levels.

In 64% of the tasks in Stein et al.'s (1996) study that remained high-level, a sustained press for justifications, explanations, and meaning, as evidenced by teacher questions, comments, and feedback, was a major contributing factor. This factor was frequently accompanied by the modelling of competent performance by the teacher or by a capable student—often in the format of a class presentation of a solution. Presentations modelled the use of multiple representations, meaningful exploration, and appropriate mathematical justification; often, successive presentations would illustrate multiple ways of approaching a problem.

Pressing for understanding is an important aspect of quality mathematics pedagogical practice that has been noted by many researchers (Kazemi & Franke, 2004). When a teacher "presses a student to elaborate on an idea, attempts to encourage students to make their reasoning explicit, or follows up on a student's answer or question with encouragement to think more deeply" (Morrone et al., 2004, p. 29), the teacher is getting a grip on what the student actually knows and providing an incentive for them to enrich that knowledge. Morrone and colleagues provide us with examples of effective teacher talk: (1) "So in this situation how did you come up with $\frac{18}{27}$ and $\frac{18}{30}$?" (2) "When can you add the way we're adding, using the traditional algorithm, finding the common denominator? When does that make sense? Several of you started, the first thing you did was add, and you ended up, what did you end up with, with $\frac{19}{16}$. What does that mean? When can you do that? When does it make sense to add that way?" (p. 33). The following vignette illustrates the way in which a teacher established a norm of mathematical argumentation with her class.

Argumentation

In a study undertaken in the US, Forman, Larreamendy-Joerns, Stein, and Brown (1998) report on a middle school teacher who had established a shared understanding of the importance of mathematical argumentation within class discussions. She guided students into the conventions of mathematical argumentation, namely, the examination of premises, the disagreement and the counter-arguments (Lampert, 1990) and made explicit their roles and responsibilities within that argumentation process. The teacher was able to skilfully orchestrate not only the discussion so that students were aligned with the academic content at hand but also guide them into particular ways of speaking and thinking mathematically (O'Connor & Michaels, 1996) Forman et al. report that in this classroom higher level thinking was fostered as students engaged in taking and defending a position that was in opposition to the claims of other students.

The teacher in the study facilitated learning for her diverse students. She shifted students' cognitive attention from procedural rules towards making sense of their mathematical experiences. They became less engaged in finding answers to the mathematical problems than in the reasoning and thinking which lead to those solutions. In Lave's (1996) words, "[b]ecoming more knowledgeable skilled [was] an aspect of [their] participation in social practice" (p. 157). By participating in a 'microcosm of mathematical practice' (Schoenfeld, 1992), they learned how and when to participate in a discursive exchange and the meaning of an acceptable or sophisticated mathematical explanation or justification.

Unlike in conventional classrooms in which "knowledge conflicts cannot emerge" (Skovsmose, 1993, p. 176), the classroom learning community relied on disagreement and conflict resolution for the negotiation of mathematical meaning. The students learned about making inferences, analysis and generalisation. Through her skill at connecting what is normally thought of as cognitive capacities, and the social and discursive grounds on which those capacities must be maintained, the teacher enabled students to put into practice the habits of mind as well as the speech and actions valued by the community of mathematical practitioners.

From Forman, Larreamendy-Joerns, Stein, and Brown (1998)

Improvements in the quality of student engagement in the IAMP was credited by [Watson and De Geest \(2005\)](#), in part, to a range of task-related strategies:

The mathematics is not simplified; there is no sense of 'finish' in the tasks, since they are stated in ways which require extended thought; there are supporting props in place (chocolate, materials, discussion); they all involve reasoning of some kind; they all contain personal challenge. (p. 39)

Improved student concentration and participation enabled tasks to be extended. The resulting sustained work on one topic promoted students' progress and awareness of progress—and hence self-esteem—that comes from being a good learner of mathematics.

In New Zealand, the formative assessment practices associated with the recently introduced National Certificate of Educational Achievement (NCEA) have the potential to affect students' task engagement. Loretz (2002), in a study of mathematics tasks used in year 11 NCEA internal assessments, found that the tasks "had moved towards high level thinking, more authentic contexts, and required more [when compared with pre-NCEA tasks] relational responses from students" (p. ii). Proponents of standards-based assessment argue that the clarity and transparency of assessment standards will reduce student anxiety, increase intrinsic motivation and self-efficacy, and promote collaboration, metacognition, and deep learning (Gipps, 1994). In particular, quality formative feedback provides students with a clear picture of what they need to do to improve (Crooks, 1988). Rawlins (in progress), working with year 12 mathematics students in three New Zealand classes, reports that students' perceptions of teacher feedback variously affected their learning strategies.

[Feedback] tells us exactly where we went wrong; how to change it and what to

do next time. My maths teacher shows us exactly where we went wrong. It's very helpful. I wish all teachers would.

When Ms. M has time, she will write working and the correct answer on our scripts. This is very helpful. Ticks and crosses are not very helpful at all and neither is just an 'm' on the front.

Ms. B is usually very good in allowing us time to read over our papers and ask any questions we have about them. I think that we need to understand what we did wrong so that we can correct it in the future. So yeah, I always use the opportunity to ask questions.

While some students appeared to want to leave the control of learning firmly with the teacher, others expressed awareness of the value of scaffolding:

I like the teacher to give me an example, her do one herself while I watch, and then I myself attempt a similar question and see if I came to the right answer. I would want her to clearly go through the method of doing it as she did her example. I don't like being left to do it on my own the first time it frustrated me.

I like to know where I went wrong and what I have to do to fix it rather than just being given the answer.

Opportunities to engage in meaningful practice activities, where the goal is to achieve understanding with fluency, are also important for learning (Marzano, 2003; Watson & Mason, 2005). Students in the IAMP study were assisted to make progress when they were given explicit guidance about 'what' they needed to remember and supported with strategies to assist them to remember.

Remembering from lesson to lesson provides continuity and a sense of short-term progress; working deeply on mathematics can aid longer-term memory, and memory for mathematics can aid deep progress and contribute self-esteem when students are aware of the extent of what they know. (Watson, De Geest, & Prestage, 2003, p. 25)

When focusing on memory with meaning, instructional strategies include reviewing areas of confusion by means of discussion in pairs, linking to work done in previous lessons, linking topics mathematically, reminding students about the central aspects of the topic, reviewing technical terms and definitions, linking words with visual or physical memory, creating concept maps, asking students to devise their own methods of recalling and testing recall, discussing memory strategies, and targeted repetitive practice. To achieve fluency, meaningful practice opportunities include significant variations each time, providing students with a sense of the range of possibilities in a topic.

To learn mathematics effectively, it is important that *all* students have opportunities to develop and participate in mathematical practices (Diezmann et al., 2004). However, the provision of tasks that are appropriate for maximising learning opportunities and outcomes for students with special needs is less clear. In Britain, Germain (2002) examined the provision for Paul, a four-year-old child with Down's syndrome, during numeracy time in a mainstream reception class. The study focused on how learning was organised for Paul and how additional adult support was used to facilitate the learning process. Although the study did not claim general findings, acknowledging that learners with Down's syndrome show a wide range of individual differences, the case study provided evidence that Paul was able to make a positive contribution to the classroom and participate in meaningful activities. His participation was supported by structural support in the form of visual clues, such as flash cards, number stamps, or stickers, and the appropriate use of ICT.

While Paul's story is a happy one, a recent review of Australasian research (Diezmann et al., 2004) suggests that students with difficulties in mathematics may be excluded through multiple channels, including reduced access to the curriculum and altered teaching approaches. Rietvelt

(2005), in her study of children with Down's syndrome in new entrant classrooms in New Zealand, found that their opportunities to learn were masked by 'busy work' and inappropriate tasks and feedback. Ian, a case study child, was frequently praised for obtaining the 'right' answer when observations indicated that he had not developed the thinking processes that would produce those 'right' answers. Another child, Mark, was observed trying to complete counting tasks that required more advanced skills than he possessed. Jonathon, the third case study child, was frequently inhibited from full engagement with the mathematical content of the task because of inappropriate peer interactions and classroom norms.

Mathematics teaching for diverse learners utilises tools as learning supports

Our theoretical framing of situated learning lends support to the understanding that when people develop and use knowledge, they do so through their interactions with the artefacts⁹ and ideas of broader social systems.

The use of specially designed artifacts is characteristic of any human activity, and of the activities of thinking and learning in particular. Most prominent in this latter category are semiotic tools such as language, specialized symbolic systems, and educational models. (Sfard & McClain, 2002, p. 154)

In mathematics education, artefacts offer 'thinking spaces'—they are tools that help to organise mathematical thinking (Askew, 2004a; Meyer et al., 2001). Symbolic artefacts or inscriptions¹⁰ characteristic of mathematics include the number system, algebraic symbolism, graphs, diagrams, models, equations, notations for fractions, functions, and calculus, and so on (English, 2002). Other tools include pictorial imageries, analogies, metaphors, models (such as pizzas, chocolate bars, and tens frames), examples, stories, illustrations, textbooks, rulers, clocks, calendars, technology, and problem contexts (Presmeg, 1992). In this section, we look at the way teachers use such resources to create an abstract or concrete frame of reference through which mathematical knowledge and procedures might be introduced, exemplified, and understood. In looking at how teachers use tools to support students' learning, we consider how students use tools to reorganise their activity.

The type of artefacts that teachers make available to students affects students' mathematical reasoning and performance. In a study involving the learning of measurement, Nunes, Light, and Mason (1993) explored the extent to which students' thinking could be attributed to different artefacts. They found that the choice of tools students can access does make a difference to their achievement. Sharp and Adams (2002) report how materials such as packs of gum, candy bars, pizzas, pumpkin pies, and orange juice—developed following discussion of suitable contexts with fifth grade students—became a powerful means to help students construct personal knowledge and establish a procedure for division of fractions. Blanton and Kaput's (2005) description of the tools that supported algebraic reasoning in a third grade classroom included objects such as in/out charts for organising data and concrete or visual artefacts such as number lines, diagrams, and line graphs for building and making written and oral arguments. When used effectively to support learning, these objects became referents around which students reasoned mathematically.

In the *Quality Teaching for Diverse Students in Schooling* BES, Alton-Lee (2003) provides evidence that teachers optimise student learning opportunities by complementing language use with multiple opportunities for students to access, generate, and use non-linguistic representations. This finding is particularly important in mathematics, where the use of students' inscriptions in the form of notations and graphical, pictorial, tabular, and geometric representations abound.

Inscriptions—the act of representing and the object itself—are an essential aspect of mathematics learning and teaching. From research with young children we have evidence

that the learning of mathematical notations is a constructive process. Five-year-old Paula's interpretation of capital numbers—"Thirty-three. So thirty is a capital number of three. And that's the other way to write the three" [pointing to the 3 in the tens place]—illustrates the interaction that takes place between conventional knowledge, such as notations, and children's invented mathematical notations (Brizuela, 2004). While both the conventions and the individual's inventions play a part in the creation of socially accepted knowledge and the making sense of mathematical conventions, researchers argue that the emphasis must be placed on the importance of children's inventions (Lehrer, Schabule, Carpenter, & Penner, 2000; Peters & Jenks, 2000).

In a New Zealand study, Warner (2003) tracked the development of year 5 and 6 students' notational schemes. She found that students' emerging ways of symbolising and notating provided a vehicle for communication, representation, reflection, and argumentation. Movement towards common notational practices was encouraged through the use of modelling books,¹¹ thinking bubbles, thinking mats and classroom norms of communication. It was noted, however, that tensions existed between formal notations and the students' idiosyncratic notational schemes. In some cases, students' attempts to adopt formal schemes acted as a barrier to the development of their mathematical understanding. Negotiating between informal and formal notation systems is a challenge for the teacher and highlights the complexity of the learning/teaching process.

Inscriptions are not limited to representations of the number system. Telling stories with graphical tools is a core element of the statistics curriculum. "A critical component in the development of students' thinking and reasoning is transnumerative thinking; that is, changing representations of data to engender an understanding of observed phenomena" (Chick, Pfannkuch, & Watson, 2005, p. 87). In order to develop statistical literacy, Chick et al. urge that students from the middle years and up need more exposure to multivariate data sets, as opposed to univariate data. These researchers argue that students should be given opportunities to create their own representations before being introduced to conventional ones, claiming that the effectiveness of standard forms may be more apparent if the students first grapple with their own representations. The following vignette illustrates how students' utilise self-generated representations to support their statistical thinking.

Telling Stories

Year 7 and 8 students, working with a set of 16 cards, are asked to look for and show any interesting features of the data. The Data Card protocol (Watson & Callingham, 1997) cards contain the name, age, weight, weekly fast-food consumption, favourite activity, and eye colour for a fictitious young person.

The researchers (Chick et al., 2005) suggest that the provision of both categorical and numerical data allows students to consider questions involving single and multiple variables and associations among variables. For example, the students who produced figure 5.8 made an appropriate transnumerative decision to calculate the average weight of the people represented in each of the 'favourite activity' categories. This represents an important first step in informal inference, where comparison of means is central to the argument and provides evidence for a difference between groups.

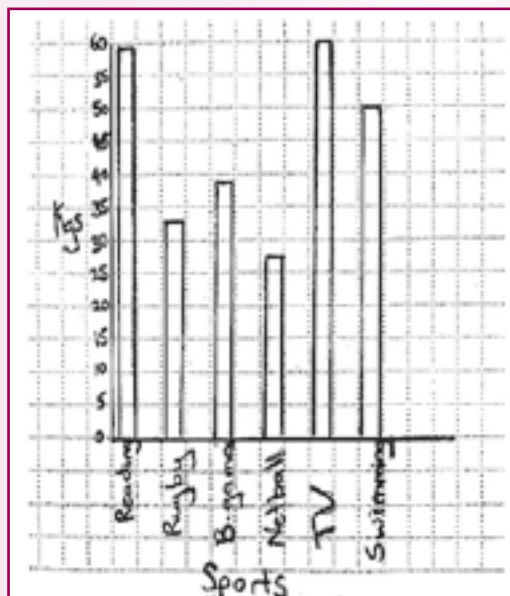
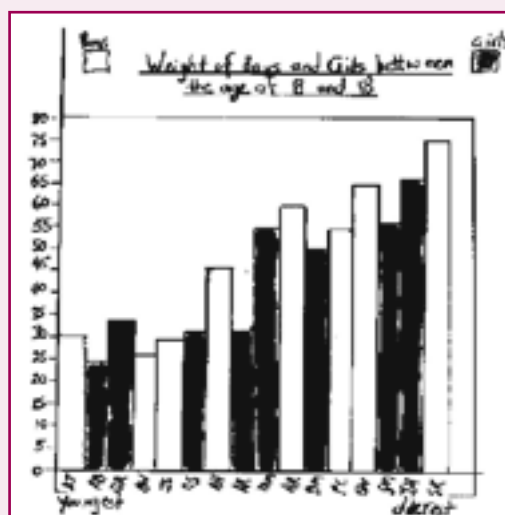


Fig. 5.8. Mean weight for each activity

When interpreting the graph in figure 5.8, the students reported surprise—they believed that the average weight for those liking swimming would be lower than for board games. This led to the development of a critical insight: "The only reason why it was high was because we didn't have enough sample ... because we only had one swimmer ... and he was quite old so he weighed quite a bit". However, despite this recognition of a third influencing variable—'age'—these students were unable to come up with a transnumerative strategy to investigate this.

According to the researchers, the capacity to transumerate and represent three variables is important, but difficult. The students whose work is shown in figure 5.9 used transumeration to incorporate the third variable of age.



What I've found

Analysis

From my graph I have discovered that boys usually weigh more than girls. This may be because of extra muscles and mass. Also older people are much heavier probably because of more muscles appearing as they mature.

Fig. 5.9. Weights are shown by ordering on age and highlighting gender

For these students, transumeration of the variable 'names of students' into a new variable, 'gender', combined with the transumeration of the variable 'age' by sorting, enabled extraction of information at a sophisticated level. The bars are used to represent weights for the ages along the horizontal axis, and colour is used to distinguish the boys from the girls. While the communication of the claim about boys weighing more than girls has not been totally successful, the increase in one numerical variable with the other is plain to see.

In each of these cases, we can see how the use of student-generated representations of the multivariate data set provided a fruitful tool to support their thinking.

From Chick et al. (2005)

Bremigan's (2005) study of 600 students' use of diagrams in an Advance Placement Calculus Examination found that the modification or construction of a diagram as part of students' problem-solving attempts was related to problem-solving success.¹² "More than 60% of students who achieved *success in set up*¹³ and over 50% of students who achieved *partial success in set up* either modified or constructed diagrams" (p. 271). Moreover, the study found that errors identified in the solutions of students who achieved partial success appeared to be, at times, related to their diagrams. For instance, students often confused the direction in which a cross-section was drawn and the related choice of model (cylindrical shell or disc) when computing the volume of a solid. Diezmann and English's (2001b) research with younger children demonstrated that tools such as 'draw a diagram' are only of value when the user is sufficiently skilled to use the tool. To address their concern that primary school students often have difficulty generating effective diagrams, Diezmann (2002) trialled an instructional programme for developing students' knowledge about diagrams (e.g., networks, matrices, hierarchies, and part-whole diagrams) and their use in problem solving. They found improved problem-solving performance for all students, noting that explicit instruction about diagrams appeared to be particularly helpful for students who have difficulty identifying the structure of a problem or are easily distracted by surface details.

Research studies (e.g., Remillard & Bryans' (2004) study of TERC material in the US) have found that, within curriculum reform programmes, teachers' orientation towards using curriculum material influences the way they use it, regardless of whether they agree with the mathematical vision embedded within the material. Different uses of the material lead to different opportunities for student and teacher learning. For example, both Ell and Irwin (2006) and Higgins (2005) have found that teachers' orientation to equipment within the New Zealand NDP was an important factor in the opportunities afforded students for discussion of mathematical ideas.

Pape, Bell, and Yetkin (2003) found that one of the features of classroom instruction that emerged as critical to students' learning was the use of multiple representations. In their seventh grade classroom study, they found that, when students were engaged in solving rich problems or creating complex representations, they were motivated and accomplished significant mathematical thinking. Multiple representations lighten the cognitive load of the learner by providing conceptual tools for thinking. Anthony and Knight (1999a) reported that New Zealand teachers of students in years 4 and 5 ranked a 'think board' as the most effective resource for promoting student understanding and remembering in early computation. The 'think board' activity, which requires students to translate equations into multiple representations of stories, pictures, symbols, and real things, supports students to make connections between different representations. Lachance and Confrey (2002) also provide evidence of the value of using concrete referents that allow students to develop an understanding of mathematical symbols at the same time as they explore the connections between various types of mathematical symbol—in this case, decimals, fractions, and percentages.

Investigations of how tools support student learning point to the critical role of the teacher. Numerous researchers (e.g., Cuoco & Curio, 2001; Gravemeijer, 1997b) have found that with teacher guidance, conceptual mediating tools can act as a springboard for discussions and for structuring mathematical knowledge. In the following vignette, we can see how, in a conceptual orientation, a tool acts as an integral aspect of the learners' mathematical reasoning rather than as an external aid to it (Cobb, 2002; Gravemeijer, 1994).

Battery Power

The setting is a seventh grade teaching experiment that focused on statistics. One of the instructional goals was that the students would come to view data as a single entity rather than as a plurality of individual data values. The instructional activities involved analysing univariate data sets in order to make a decision or a judgment with the support of a computer-based analysis tool. The tool was designed to enable students to structure the data in a way that fitted with their current ways of understanding while simultaneously building toward conventional graphs.

One of the initial task situations involved comparing two separate brands of batteries, Always Ready and Tough Cell, based on a given data set of a sample of ten batteries of each of the two brands that had been tested to determine how long they would last (see fig. 5.10).

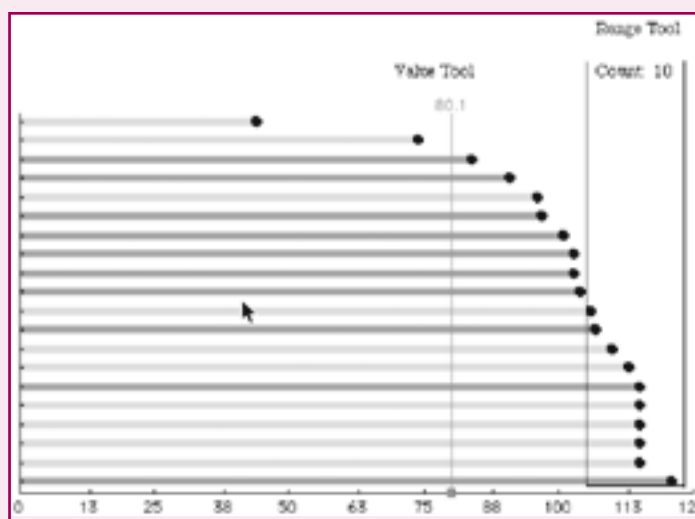


Fig. 5.10. Data set of two types of batteries

Using the computer minitool, the students were able to use the 'Value' and 'Range' functions to explore the data and make claims as to which brand was better. The ensuing justifications and meaning-making appeared to be related to the way in which individual students used the tools to support their thinking.

Celia was the first student to share her argument. She began by explaining that she used the range tool to identify the top ten batteries out of the twenty that were tested. In doing so, she found that seven of the longest lasting were Always Ready batteries. During the discussion of Celia's explanation, Bradley noted that he compared the two brands of batteries a different way.

Bradley: Can you put the representative value of 80? Now, see there's still [Always Ready batteries] behind 80, but all the Tough Cell is above 80 and I'd rather have a consistent battery that is going to give me above 80 hours instead of one.

Teacher: Question for Bradley? Janine?

Janine: Why wouldn't the Always Ready battery be consistent?

Bradley: All your Tough Cells is above 80 but you still have two behind in the Always Ready.

Janine: Yeah, but that's only two out of ten.

Bradley: Yeah, but they only did ten batteries and the two or three will add up. It will add up to more bad batteries and all that.

Janine: Only wouldn't that happen with the Tough Cell batteries?

Bradley: The Tough Cell batteries show on the chart that they are all over 80, so it seems to me they would all be better.

Janine: [nods okay].

Bradley based his argument on the observation that all the Tough Cell batteries lasted at least 80 hours. Using the value bar to partition the data, he determined that Tough Cell was a more consistent brand. In contrast, Celia's used the range tool to isolate the ten longest lasting batteries. As the students reasoned with these tools, new meanings emerged (e.g., batteries of a particular brand are better because more of those batteries last the longest; batteries of a particular brand are consistent because all last over 80 hours).

As the discussion proceeded, Celia's choice of the 'top ten' was open to question. For instance, one student pointed out that if she had chosen the top fourteen batteries instead of the top ten, there would be seven of each brand. Celia's choice of the top ten was arbitrary in the sense that it was not grounded in the context of the investigation. Bradley, however, gave a rationale for choosing 80 hours that appeared to make sense to the students. He wanted batteries that he could be assured would last a minimum of 80 hours. As a consequence of his argument, this position was then accepted as valid.

The students continued to use the value bar to partition other data sets in this way. The researchers note that their expectation when they designed the tool was that students might use the value bar to 'eyeball' that centre of balance point of the data set. Instead, students used the value tool to partition data sets and to find the value of specific data points; they adapted the minitool to match their current ways of thinking about the data.

From McClain, Cobb, and Gravemeijer (2000)

Young-Loveridge (2005) illustrated the way in which one teacher used conceptual mediating tools to both advance students' learning and redefine the teacher–student power relationship. The investigation of the teacher's pedagogical practices took place within the context of a year 5 and 6 classroom. The class had been involved with the New Zealand NDP and the teacher had also been working on developing student-centred assessment tools. The particular tool that the teacher used for enhancing student learning took the form of 'learning logs', in which the teacher wrote a comment about an individual's learning to initiate student thinking. The learning log was then used by students to record their own learning intentions and for assembling evidence of student work that exemplified how students had met those intentions. The teacher noted that "[w]hat the learning logs have sparked for us really is the importance of the teacher–student relationship and the power that teachers have traditionally held over students, and the ways we've been breaking that down ..." (p. 111).

A number of researchers (e.g., McClain & Cobb, 1999) support the usefulness of having students make written records of their work. Ball (1997) describes students' written work as an artefact of teaching and learning that provides "promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking" (p. 808). Teacher knowledge can also be enhanced: teacher development projects (e.g., Davies & Walker, 2005; NDP, 2004) provide evidence that teachers who engage in discussions about students' work and mathematical thinking develop their own mathematical awareness.

Unlike 'learning logs' as used in Young-Loveridge's study, the school mathematics text book is a familiar tool for structuring students' knowledge. Grouws, Cooney, and Jones (1988) argue that the textbook is the most important influence on students' attitudes towards mathematics. Goos (1999) vividly illustrates the way in which a year 11 teacher provided support for a class to critically engage with the ideas in a mathematics textbook. Goos observes that the readings were both assigned and spontaneous. The teacher had planned a linear process of whole-class discussion following on from the assigned reading. The students proceeded iteratively in their quest for understanding of the concepts elaborated in the reading. They read and asked questions of each other and the teacher, moving back and forth from reading to discussion in order to make sense of the reading. But it was the teacher's guidance that allowed the students to articulate in appropriate and acceptable mathematical language their conceptual understandings of the worked examples in the textbook.

In their study of students' growth in mathematical understanding, Pirie and Martin (2000)

note that a textbook or written notes may act as an important tool for students (especially at the secondary level) who are involved in what they term ‘collecting’. Collecting occurs when students know what is needed to solve a problem but don’t have sufficient understanding to be able to automatically recall useable knowledge. The researchers contend that the teacher can actively support students to fold back and collect by overt modelling of ‘collecting’ when working examples, by promotion of writing about one’s understanding, by assistance with reading texts, and through student discussion and direct intervention—for example, reminding students of a particular technique in order to allow them to make progress in the building of a new concept.

Lerman (1993) records that textbooks have been shown to play a role in reflecting real-life conditions in the classroom and in reproducing existing social values. Social messages are sometimes conveyed implicitly, and the teacher who facilitates learning for diverse students is able to critically examine and challenge the assumptions on which those messages are based. Lerman points out that the most widely-used textbook series in British schools caters for different ability levels through different texts. It conveys arguable social messages. For example, while the textbook that targets top-streamed students asks them to calculate tax on an income of £50,000, the version that targets lower-ability students asks them to calculate tax on an income of £9,000. The implicit assumption is that low mathematical ability will predestine a student to a low-income future.

In the New Zealand context, Higgins et al. (2005) found texts to be an important influence in structuring teachers’ own knowledge of numeracy. The teacher of an English-medium class comprising mainly Māori students used the teachers’ manual (available to teachers involved in the NDP) to guide the way in which she shaped learning in the classroom. The supporting framework of the manual, combined with the pedagogical method advocated in the NDP, allowed this teacher to be more responsive to her students.

Like the teachers’ manual, the concrete equipment used in the NDP plays a part in structuring knowledge. Higgins (2005) found that equipment allowed students to experience mathematical operations and relations at first hand. Through the mediating role played by equipment, new concepts were introduced, images evoked, and discussion initiated before the concepts were completely understood. However, in an earlier study, Higgins (2001) found that some teachers believed that the use of number equipment was more suitable for children in lower academic groupings. The use of equipment was also strongly linked to the notion that some students are kinaesthetic learners—a label frequently attached to Māori and Pasifika students. In the *Quality Teaching BES*, Alton-Lee (2003) contends that differential expectations for, and treatment of, Māori and Pasifika students on the grounds of supposed ‘learning styles’ can have negative outcomes for some students.

Teachers who foster students’ mathematical development make continual inferences about the way their students ‘see’ the mathematical concepts embodied in the artefacts used. Reliable inferences, however, can only be made from appropriate external representational choices (English & Goldin, 2002). Ball (1993) and Lampert (1989) have found that effective teachers select and construct artefacts that their students can relate to and have the intellectual resources to make sense of and then extend their students’ capacity to reason with and about the ideas under scrutiny. Walls (2004) looked at the specialised equipment, such as number lines, number tracks, and flip boards, utilised in the New Zealand Numeracy Project. She questions whether these adult-contrived artefacts are as effective for students as the familiar objects used in everyday discourse. She contrasts the number equipment with that used within The Realistic Mathematics Education (RME) Programme of the Netherlands. In his work in the RME, Gravemeijer (e.g., 1997a) has shown that through a process of generalising and formalising, meaningful equipment gradually takes on the form of its own and contributes to the shaping of mathematical reasoning. Walls makes the suggestion that New Zealand classrooms might use real life examples such as “maramataka [traditional Māori calendar], thermometers, elevator floor tracking systems, clocks, microwave timers, game boards, speedometers, and navigational

equipment” (p. 31). These conceptual tools allow students to develop facility with multiple and directional counting and “flexibility in understanding numbers sequenced in tracks that are presented in a wide range of layouts from right to left, left to right, top to bottom, bottom to top, etc.” (p. 31).

Baxter, Woodward, and Olson’s (2001) study in five elementary schools in reform mathematics programmes within the US noted differences among the classes in terms of the mathematical role that manipulatives played. Observations of 16 low-achieving target students revealed that whilst in some classes manipulatives were a distracter, in others they provided a conceptual scaffold. In three of the five classrooms, manipulatives became the focus rather than a means for thinking about mathematical ideas. For example, when working in pairs with fraction bars, Ginger, a target student, kept the materials in neat piles. Her partner worked on the mathematical task and simply asked Ginger for particular bars. Although the fraction bars served to involve Ginger in the task they did not appear to further her understanding of relationships among fractions. In contrast, a distinctive feature of instruction for those teachers who engaged target students in mathematical thinking was the way they used a variety of representations of a concept prior to the use of the manipulative specified in the curriculum. For example, in a geometry lesson, parallel lines were represented by a range of arm movements, lengths of string were used to create angles, calculators were used, and finally representations were transferred to geoboards. All students, including those easily distracted, worked with a wide array of geometric terms, building conceptual understandings of important mathematical ideas, such as ‘parallel’, rather than memorising a list of definitions generated by the teacher.

Research has shown that tools can provide effective compensatory support for students with learning disabilities. Jones et al.’s study (1996, cited in Thornton et al., 1997) illustrates how a nine-year-old learning-disabled student, Jana, used the 100s chart to move beyond pencil-and-paper computation. The episode took place within a series of lessons structured around calculations relating to ‘garage sale’ purchases.

Jana spoke for herself and her partner: “We picked the picture frame for 38c and the poster for 15c—and we just have 7c change.” When asked to explain how they know they would have 7c change, Jana said, “We just thought about the 100s Chart. We started with 38 and went down to 48 and then counted 5 more. So we paid 53c—that gives us 7c back because we had 60c to spend.

During early instruction with this graphic aid, Jana was encouraged to move a finger along the chart as she counted. With practice, she developed the skills to visualise the counting-on process just by thinking of the 100s chart—using the chart as a compensatory tool to compute two-digit sums mentally.

Recent research has focused on the students’ use of models. Models, such as drawings, diagrams, stories, or formal-symbolic representations, can be seen as a tool for bridging the gap from concrete situations to abstract mathematics—thus acting as “a tool for strategic thinking” (van Dijk, van Oers, Terwel, & van den Eeden, 2003). However, the meanings of models developed by teachers are not always readily apparent to students—there is a danger that a ready-supplied model attempts to transmit an adult’s way of thinking. The following vignette from van Dijk et al.’s experimental research study with 10 classes of grade 5 students considers how best to support students’ mathematical structuring activity.

Coffee Pots

The curriculum involved open, complex problems¹⁴ developed for percentages and graphs. The intervention consisted of 13 lessons: one lesson in which students learned about strategy use, models and their functions, and 12 lessons on percentages and graphs. In the experimental condition students co-constructively were encouraged to design their own models, as a tool for the learning of percentages and graphs. In the control condition students learned to apply ready-made models provided by the teacher. They did not learn to design or choose models themselves. For example, in the experimental condition students were asked to invent representations (designing condition) of given percentages, whereas in the control condition the students were asked to recognise percentages and shade this amount on a given model (provided condition). In both cases there was a recognition that percentages and fractions are conceptually interwoven, meaning that there was explicit emphasis on the connections between fractions and percentages.

In the following abbreviated excerpt from an experimental class, several models were presented on the blackboard for discussion. Each of the models was meant to represent a coffeepot (that holds 80 cups), which was 25% filled with coffee.

Bart: I drew a coffeepot.

Teacher: Ok, and how can we see it is for 25% filled with coffee?

[Some further discussion about 25%, 50% and 75% reference points.]

Ann: I made a sort of thermometer, and I put 100 small lines in it. And then I coloured the first 25 lines red.

Teacher: OK, and how did you figure out how many cups of coffee this pot contains?

Ann: I knew that 25 out of 100 is a quarter.

Teacher: Can you show it on your model? [Anne points at her model.]

Anne: Well, I knew that the total pot contained 80 cups, and then I took a quarter out of 80.

Teacher: Good. Did it take you long to draw this model?

Anne: Uh ... yes, quite long ...

Teacher: Who can think of a solution to make this model easier to draw?

Jesse: I would not draw all those 100 lines, but only the most important ones, like 0% and 100%.

Teacher: OK, that would make it much easier. Linda, can you explain your model?

Linda: I just made a kind of a bar. That's supposed to be the coffeepot. And then I shaded the first quarter of it.

Teacher: Why did you choose a bar to represent the coffeepot? It doesn't look like a coffeepot at all!

Linda: Because the model doesn't really have to look like a coffeepot. I think a bar like this is just easy to draw. And you can use it in every situation.

Teacher: Well we saw three models ... What model do you like best? ... Why?

Jim: Linda's, because it's fast, you don't need to draw the whole coffeepot. And you don't need to draw 100 lines, like in Ann's models.

The teacher then summarises the advantages of each of the models.

In contrast to the control group, in which the discussion revolved around one given model, the designing group offered various models. The invitation to explain their model and their thinking to other peers provided opportunities for others to suggest improvements and the teacher to ask critical questions to stimulate reflection to provoke improvements. The group as a whole learned about the process of model designing, and how to use models to solve math problems.

The study concluded that the strategic learning of 'how to model mathematical problems' was

beneficial to student problem solving performance. Students who learned to construct models in the experimental programme scored significantly better on the post-test than students who learned to work with models provided by the teacher. For the post-test, the effect size was .4. Although the researchers admit that no decisive, quantitative proof is given, they make a strong case that there is convincing evidence from qualitative data to support their belief that “designing models in co-construction may lead to students developing deeper insight into the meaning and use of models and consequently make possible a more flexible approach in problems solving” (p. 184).

From Van Dijk et al. (2003)

Concrete mediating tools both empower and limit mathematical thinking. Research has shown that even when tools maintain a high degree of fidelity to the problem situation, no representational context is perfect. A particular representation “may be skewed toward one meaning of a mathematical idea, obscuring other, equally important ones” (Ball, 1993, p. 162).

Inscriptions do not merely copy the world; they select and enhance aspects of it, making visible new features and relations that cannot be seen by observing the objects and events themselves. For example, a road map selects and enhances aspects such as distance relationships and scale that are not visible to an individual at ‘ground level,’ while leaving out other features that are not important to the purposes of the inscription—such as trees, power lines, and buildings. (Lehrer, Schauble, & Petrosino, 2001, cited in Sfard & McClain, 2002, p. 155)

In considering the use of concrete artefacts, Askew and Millett (in press) note the possibility that students and teachers will give them different interpretations. “Being aware of the strengths and weaknesses of various artifacts is linked to subject knowledge. Simply convincing teachers that certain artifacts—number lines, counting sticks—are good things to use in lessons is not sufficient.” Whereas teachers might ‘see’ the mathematical concept embodied in the representations, the student may see nothing more than the concrete material.

As noted earlier, classroom representational contexts “should be real or at least imaginable; be varied; relate to real problems to solve; be sensitive to cultural, gender and racial norms and not exclude any group of students; and allow the making of models” (Sullivan, Zevenbergen, & Mousley, 2002). A number of researchers provide evidence that, despite assumptions that meanings are shared, conceptual tools may actually impose different conceptualisations of apparently purely abstract mathematical problems (e.g., Masingila, 1993; Scribner, 1984). Tools or symbol systems may structure the learning system but can have different meanings for different cultural groups; the same artefact may be understood differently by different cultures (Nunes, 1997).

Authentic situations make it possible for artefacts to provide a bridge between the mathematics and the situation. Lowrie (2004) found that children involved in a planning activity (costing and scheduling a family excursion to a theme park) were assisted to ‘make sense’ of the task through the use of brochures, menus, bus timetables, and photographs. Students were observed to extend, adapt, and revise mathematical ideas within a personalised context. Often the problem modification was strongly influenced by personal experiences unlikely to be considered in typical problem-solving activities in this classroom. Students established their own sense of authenticity by aligning the problem with their personal experiences and understandings: “They have built more rides since I was there.” [Sue]; “I didn’t like that you had to have 10 minutes between each ride ... because the last time I was there I only took a few minutes to walk from one ride to another” [Stuart]. The cultural artefacts provided an opportunity for all of the participants to engage in a problem scenario that was contextually meaningful. Significantly, some of the children who were not considered ‘mathematically capable’ invented more powerful ideas than those who did not see the task as an open-ended challenge.

As noted in earlier discussion on contextual tasks, it is a challenge to know which situations and related artefacts will appeal to or motivate learners. For some, the type of contextual material matters. Mack (1993) offered a comment from a student in her fraction study: “I don’t

like pizza so I never eat it. I love ice cream and eat it everyday, so if you make the problems ice cream, it'll be easier for me." (p. 99)

As well as supporting thinking, tools provide an effective way for students to communicate their thinking. For example, Hatano and Inagaki (1998) describe an instructional episode involving first grade children in a Japanese classroom. Students familiar with join–separate problems were presented with the problem: *There are 12 boys and 8 girls. How many more boys than girls are there?* Most of the children answered correctly that there were four more boys, but one child insisted that subtraction could not be used because it was impossible to subtract girls from boys. None of the students who had answered correctly was able to argue persuasively against this assertion. It was only after the students physically modelled the situation that they realised that finding the difference was a matter of subtracting the 8 boys who could hold hands with girls from the 12 boys.

Nunes and Moreno (2002) describe a successful intervention programme for deaf children in London. The programme consisted exclusively of activities involving mathematical reasoning—as opposed to more traditional programmes consisting of teaching of algorithms. Particular emphasis was placed on visual means—such as tables and graphs—of representing relations between variables. However, because it was difficult to identify the crucial cognitive lever in this project, the researchers recommend further research to establish whether the use of visual means of communication alone can produce such positive results. They suggest that such approaches may well benefit many hearing children also.

Technological tools

How can technology be used as a resource to develop mathematical knowledge? Technological tools, like other conceptual mediators, can act as catalysts for classroom collaboration, independent enquiry, shared knowledge, and mathematical engagement. Reporting on a rich multi-dimensional task on the topic of Energy, which incorporated the use of ICT tools, Yelland (2005, p. 237) noted that opportunities for mathematical exploration included:

- looking at chronological order in sequences of *time*;
- incorporating *time* and *measurement* concepts into the editing feature of *i-movies*;
- science experiments involving *capacity*, designed to explore water consumption and conservation;
- *counting, ordering, and using number operations* for organising information for presentation to the group;
- investigations involving *money* in order to discover more effective ways of using energy such as electricity and gas;
- discussions of *space and shape* when considering, for example, storage of petroleum, transmission of electricity, and the structure and shape of electricity pylons.

Yelland notes that the children articulated their enjoyment of this project work in a variety of ways: they liked working with their friends, choosing what to do, and using computers to ‘find out stuff’ on the Internet and to make (PowerPoint) presentations and movies.

In the secondary environment, Arnold (2004) found that algebraic tools available on a computer not only offer mathematical insight but also make students’ tacit mathematical understanding public. Likewise, Goos, Galbraith, Renshaw, and Geiger (2000) provide evidence that the graphics calculator can be a catalyst for personal (small-group) and public (whole-class) knowledge production. The calculator operates as conceptual mediator in these ways:

- message stick—students pass their calculators back and forth to compare their working and solutions;
- scratch pad—students make changes to both their own or others’ work on calculators;

- partner and collaborator—students speak directly to their calculators as collaborating partners, articulating the key functions and menu choices.

The research of Goos and colleagues (2000) provides an example of how the intellectual resources of the teacher, taken together with the mediating properties of spreadsheet, function-plotting software, and graphics calculator, can contribute to the production of meaningful student knowledge about mathematical iteration. In his discursive interactions, the teacher focused on mathematical thinking. In particular, his support centred on suggestions that the students might explore both the spreadsheet and the graph, the issuing of specific challenges, suggestions that they consult with others in the class, and invitations to open their findings to class scrutiny.

In the New Zealand context, Thomas and colleagues have conducted several classroom investigations into the use of graphics calculators by secondary students. In a study that set out to compare the teaching of variables in algebra with and without the use of graphics calculators, Graham and Thomas (2001) found that students in the intervention group exhibited improved understanding of the use of letters for specific unknowns or generalised numbers. In discussing the value of the graphics calculator as a tool, Graham and Thomas argue that it:

helps to build a versatile view of letters because the physical experience of the students is that of tapping keys to place different numbers into the calculator stores, changing the values in those stores and retrieving values from them. In this way they personally experience that letters can represent numbers; that they can have more than one value, albeit at different times; and that their value can be combined with other numbers or the value of other letters. Hence they build the idea of letter, not as a concrete object, but as a placeholder for a number or a range of numbers; as a procept. (p. 278)

In a study involving year 11 students, Hong and Thomas (2001) found that use of CAS (computer algebraic systems) calculators helped students make connections between sub-concepts across representations in a calculus unit involving the application of the Newton-Raphson method. The researchers claim that a graphics-oriented approach to the learning of calculus can facilitate more challenging problems, with less focus on procedural skills, formulas, routine problems, and exercises. While remaining convinced of the value of graphics calculators to help students develop deeper understanding of functions, Thomas and colleagues note that research is needed to find a suitable pedagogical format and to ascertain the extent of any affective or social components to the advances in learning demonstrated in these studies.

Technological tools can serve to increase the relevance and accessibility of mathematical practices for learners (Goos & Cretchley, 2004). Nason, Woodruff, and Lesh (2002) report on a study of a sixth grade classroom, in which groups of students developed spreadsheet models to record quality of life in a number of Canadian cities. In an iterative process that began with group members' agreements on criteria for ranking cities, followed by group presentations to the class and visual investigations into each others' spreadsheets, the students produced models that allowed them to rank the cities. As part of their study, the researchers explored the potential of the computer to stimulate collaborative student efforts. As a result of public and critical scrutiny of their ideas, the students learned about mathematical efficiency and organising information for presentation. The computer became a mediator not only for building personal knowledge but also for the development of learning at the interpersonal level. It did this by occasioning interactions within and between student groups in the classroom.

Engaging with technologies can change or challenge traditional student learning trajectories. Herbert and Pierce (2005) used computer-generated modelling to challenge the traditional approach to teaching rate by studying the gradients of linear functions. They argue that the constant rate of change of linear functions does not prepare students for the complexity found in situations where the rate varies.

Rate of Change

Herbert and Pierce (2005) investigated year 10 students' use of MathWorlds¹⁵ to explore multiple representations of the simulated movement of a lift in a multi-storey building. The experientially real context for students provided a bridge between their informal and formal knowledge, leading to a concept image of rate of change which is mathematically correct and potentially useful for further study in calculus. The movement of the lift is represented by a position-time graph, a velocity-time graph, a numeric display, a table and an algebraic rule. The interconnectivity of control between the multiple representations of the movement of the lift enables real-time investigations. For example, changing the symbolic representation of the situation results in corresponding changes in the animation, graphs and tables. Moreover, the movement of the lift is controlled by its representations and the representations of its movement are controlled by the animation.

Pre- and post-tests and interviews provided clear evidence of students' development of contextually appropriate rate-related reasoning. Post-test interviews suggested that students' concept image of rate of change was enhanced with rate-related reasoning rather than formulaic calculations of gradients used in problem solutions. For example, in the following responses it is evident that the students were thinking in terms of change in one variable per unit change in the other rather than calculating a gradient with the use of a formula.

Student A: ... plus 3 each time for the number of dollars for each bag.

Student B: It goes up by 3 for every bag bought.

Student C: ... the graph will go up \$10 for each hour that he's worked, so say for 1 hour it would be up here at 30—it jumped up to 30 in the first hour and the second hour it would go to 40.

The researchers claim that the use of the motion context of lifts in MathsWorlds to develop numerous 'models of' rate of change facilitated the development of an emergent model for rate of change in other contexts. In this way velocity has become a 'model for' rate of change to solve problems in non-motion contexts thus expanding students' concept image of rate of change.

From Herbert and Pierce (2005)

Dynamic software, with its provision of a rich visual environment, can also act as a catalyst for more abstract geometric reasoning. In a year 8 Australian classroom, Vincent (2003) found that where students worked in pairs on an exploratory task using the dynamic software Cabri Geometry, there were improvements in students' arguments and their ability to connect conjecturing with proving.

Spreadsheets, an everyday software application, have also proved to be a useful mathematical tool. Calder's (2005) analysis of children's investigations illustrates how the use of spreadsheets shaped a number pattern investigation. The availability of the spreadsheet led to the children familiarising themselves with and then framing the problem through a visual, tabular lens. For example, when investigating the pattern formed by the 101 times tables, attention was focused on the discrete visual elements of the pattern rather than the consequence of an operation. However, Calder noted that while this particular medium opened up unique avenues for exploration, it simultaneously fashioned the investigation in a way that may have constrained the understanding of some children.

New technologies are not limited to computers and calculators; increasing numbers of students have access to a range of digital and mobile technologies. Davies and Hyun (2005) studied a group of 18 culturally and linguistically diverse children (aged five and six) and mapped their development of spatial representation through map-making activities. Using digital cameras, video imaging, and sketchbook drawings, the children captured real images of the area under study and downloaded the images into the computer. The children demonstrated innovative and productive problem-solving approaches as they negotiated ways to present familiar spaces. Currently, several New Zealand secondary school clusters¹⁶ are involved in classroom-based

research studies investigating the integration of data obtained from probes (heat, motion, and light sensors) with mathematical activities.

Although increased access to the Internet is changing our everyday lives, Goos and Cretchley (2004) note that this is an area where technological advances are outpacing research into their implications for education. In their review of mathematics education research in Australasia for the period 2000–03, they conclude that “there is an ongoing need to investigate how technology-mediated communication and course delivery might lead to different kinds of teaching–learning interactions or add value to students’ learning” (p. 168).

While new technologies provide an opportunity for pursuing purposeful mathematical tasks, research has found that it is their teachers’ pedagogical practice that influences the way in which technological tools are used by students. Ball and Stacey (2005) suggest that teachers should share decision making about mental, pencil-and-paper, and technology-based approaches with their classes and have the students monitor their own underuse or overuse of technology. Particularly for secondary students, the use of CAS technology needs to be accompanied by the development of algebraic insight, including the ability to identify the structure and key features of expressions and to link representations. When students see an algebraic expression, they should think about what they already know about the symbols used, the structure and key features of the expression, and possibly its graph before they move further into the question. Pierce and Stacey (2004) have found that when teachers routinely demonstrate this initial step in class, it is likely to become a habit for their students.

Assessment practices, especially those involving examination questions related to the nature of algebraic and calculus procedures, are also affected with the introduction of CAS. In reviewing assessment practices in calculus examinations, Forster and Mueller (2002) note that the presence of graphics calculators requires increased awareness on the part of the teacher of the balance that needs to be maintained between opportunities for visual, empirical, and analytic approaches when solving tasks.

Pedagogical practices associated with access to resources will be increasingly affected as teachers move to incorporate technology-based presentations and web-based facilitation of learning (Heid, 2005). McHardy (2006) reports on an action research study investigating the utilisation of PowerPoint presentations and on an email discussion group with NCEA level 3 Mathematics with Calculus and Mathematics with Statistics classes. Students reported that the email contact and availability of online resources supported their learning by increasing their access to information and by giving them a more flexible work environment and greater opportunities to practise. This study provides a first glimpse at the potential and possible impacts of greater integration of technology and pedagogical practices in New Zealand mathematics classes.

Awareness of students’ attitudes towards technology continues to be a central concern in evaluating the impact of technologies on mathematics learning. From recent surveys with Australian secondary students, we find mixed views. Reporting on one senior class group, Geiger (2003) notes that the students remained equivocal about the extent to which using technology to solve problems helped them understand the underlying mathematics. Reporting on two junior secondary classes, Vale (2003) found that boys reported more positive attitudes than girls towards learning mathematics with computers. These findings match the gendered patterns of behaviours that were observed when the students worked on computer-based mathematical tasks. In a much larger study, Forgasz (2002) explored the beliefs of lower secondary students in 28 schools. She found that the most traditional, gender-stereotyped beliefs about computers were held by students from higher socio-economic backgrounds or those attending higher socio-economic status schools.

Increasingly, research suggests that effective pedagogical support needs to be cognisant of the affective and social aspects of student use of technologies. For example, in a study of CAS technology, Pierce, Herbert, and Giri (2004) found that where teachers continue to privilege the high value of done-by-hand algebraic manipulations, students are unlikely to become effective

users of CAS. Students in this study perceived that CAS offered insufficient advantages over a graphics calculator to warrant the time and cognitive effort required to become effective users of this new technology. The researchers found that messages conveyed by teachers' words and actions were of paramount importance in realising the potential gains of CAS:

[U]nambiguous value should be given to the alternative solution methods afforded by CAS. In particular this means rethinking what is meant by 'showing steps' or 'showing working' and the consequences for making. New, acceptable standards need to be established and clearly communicated to students if they are to be expected to work towards developing effective use of CAS. (p. 469)

Whatever the form of tool, quality teaching requires careful consideration of the purpose of its use, how it will be valued, and whether the outcomes are justified by the learner investment required.

Conclusion

For all students in the mathematics classroom, the 'what' that they do is crucial to their learning. The 'what'—in terms of the task set by the teacher—will be mediated by a complex array of factors, some determined by the individual student's knowledge and experiences, and others by the pedagogical affordances, constraints, and participation norms of the classroom.

Teachers' pedagogical content knowledge plays a central role in their planning and organisation for mathematics instruction. Teachers' normative ways of reasoning with instructional materials necessarily encompass both the mathematical domain that is the focus of instruction and the diverse ways students might approach and solve instructional activities. When setting tasks, teachers need to ensure that they are designed to support mathematics learning in the first instance—that they are appropriate and challenging for all students. The provision of low-level tasks accompanied by low-level expectations can limit students' mathematical development. A proficiency agenda, as advocated by Watson (2002), "does not dwell simply on the positive aspects of behaviour, motivation or attitudes, although those would play a part, it would also recognize and emphasize thinking skills which students exhibit and offer opportunity for these to be used [by all students] to learn mainstream curriculum mathematical concepts" (p. 473).

The research provides evidence that tasks vary in nature and purpose, with a range of positive learning outcomes associated with problem-based tasks, modelling tasks, and mathematics context tasks. But whatever their format, effective tasks are those that afford opportunities for students to investigate mathematical structure, to generalise, and to exemplify.

The opportunity for learning also rests with what students themselves are helped to produce. The effective use of instructional activities and tasks, alongside other resources and tools, enables students' mathematical reasoning to be visible and open for reflection. Students' own ideas are resources, both for their own and others' learning; their representations stimulate others' thinking, and their explanations challenge and extend. Support for significant mathematical thinking is dependent "on teachers who can hear the mathematics in students' talk, who can shape and offer problems of an adequate size and sufficient scope, and who can steer such problems to a productive point" (Bass & Ball, p. vii).¹⁷

¹ The researchers noted, however, that while a strong mathematics curriculum, more extensive professional development and teacher support, and a whole-school reform mode made an important difference, the achievement gains were "not large enough to meet current accountability expectations under NCLB" (p. 58).

² Currently Mulligan and colleagues are engaged in a professional development programme aimed at developing teacher's pedagogical knowledge and children's use of pattern and structure in key mathematical concepts.

³ The researchers Blanton and Kaput take 'algebraic reasoning' to be "a process in which students generalise mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (Blanton & Kaput, 2005, p. 413).

- ⁴ Note: Askew interprets $? \times 5$ as ‘something multiplied by 5’ not ‘something times 5’.
- ⁵ Similarly the literature discusses ‘ill-structured’ problems—problems that “lack a clear formulation, or a specific procedure that will guarantee a solution, and criteria for determining when a solution has been achieved” (Kilpatrick, 1987, p. 134).
- ⁶ Difficulties would occur where the convention is to read $X \times Y$ as X groups of Y .
- ⁷ See Sfard, A. (2005). What could be more practical than good research? *Educational Studies in Mathematics*, 58, 393-413.
- ⁸ The Dutch verb *zich realise-ren* means to imagine.
- ⁹ In general the terms, artefacts and tool are used somewhat interchangeably in the literature. In a special issue of *The Journal of the Learning Sciences* (2002, Vol. 11) tools in the mathematics classroom are regarded as ‘designed artifacts’.
- ¹⁰ Some researchers more commonly use the term ‘representations’.
- ¹¹ For further discussion of the role of ‘modelling books’ in NDP classrooms see Higgins, J. & Wakefield, M. (in preparation). *Modelling books and students’ discussion in mathematics*
- ¹² The researchers note that the methodology used in their study meant that it was impossible to detect if or how students who did not modify or construct a diagram used other visualisation strategies, such as the creation of mental images rather than physical images, or how they may have used the given diagrams without making markings.
- ¹³ Success in ‘set up’ indicated that students were able to construct an algebraic expression, equation, or integral, which, when solved or evaluated, would lead to the problem’s solution.
- ¹⁴ Problems were selected from the current maths methods course used in The Netherlands and in the US (Mathematics in Context) and exercises that had been developed for MILE (Multimedia Interactive Learning Environment), a project of the Freudenthal Institute.
- ¹⁵ MathWorlds is an animation software base on those obtained from the SimCalc website.
- ¹⁶ Numerous workshop presentations on the application of technologies were presented at the September 2005 New Zealand Association of Mathematics Teachers Biennial Conference including a report on the current Mobile technology in the Sciences (MOTIS) project by A Tideswell.
- ¹⁷ Foreword in Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

References

- Ainley, J., Bills, L., & Wilson, K. (2005). Purposeful task design and the emergence of transparency. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 17-24). Melbourne: PME.
- Alexander, R. (2000). *Culture and pedagogy: International comparisons in primary education*. Oxford: Blackwell Publishers.
- Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington: Ministry of Education.
- Anderson, D., Averill, R., Easton, H., & Smith, D. (2005). Use of a cultural metaphor in pre-service mathematics teacher education. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 81-88). Sydney: MERGA
- Anderson, J. (2003). Teachers’ choice of tasks: A window into beliefs about the role of problem solving in learning mathematics. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 72-79). Sydney: MERGA.
- Anghileri, J. (2002). Scaffolding practices that enhance mathematics learning. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49-56). Norwich: PME.
- Anthony, G. (1994). *Learning strategies in mathematics education*. Unpublished doctoral dissertation, Massey University, Palmerston North, NZ.

- Anthony, G. (1996a). Classroom instructional factors affecting mathematics students' strategic learning behaviours. In P. Clarkson (Ed.), *Technology in mathematics education* (Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia pp. 38-48). Sydney: MERGA.
- Anthony, G. (1996b). When mathematics students fail to use appropriate learning strategies. *Mathematics Education Research Journal*, 8(1), 23-37.
- Anthony, G., & Hunter, R. (2005). A window into mathematics classrooms: Traditional to reform. *New Zealand Journal of Educational Studies*, 40(1 & 2), 25-43.
- Anthony, G., & Knight, G. (1999a). *Teaching for understanding and memory in Year 4 and 5 mathematics*. Palmerston North: Massey University.
- Anthony, G., & Knight, G. (1999b). The role of practice in mathematics learning. *The Mathematics Educator*, 4(2), 91-107.
- Anthony, G., & Walshaw, M. (2003). *Fractions assessment using a real context* (NEMP Probe Study Report). Palmerston North: Massey University.
- Arnold, S. (2004). Classroom computer algebra: Some issues and approaches. *Australian Mathematics Teacher*, 60(2), 17-21.
- Arnold, S. (2005). Investigation functions: Exploring calculus concepts. *Australian Senior Mathematics Journal*, 19(1), 50-56.
- Askew, M. (1999). It ain't (just) what you do: Effective teachers of numeracy. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 91-102). Buckingham: Open University Press.
- Askew, M. (2003). Word problems: Cinderellas or wicked witches? In I. Thompson (Ed.), *Enhancing primary mathematics learning* (pp. 78-85). Buckingham: Open University Press.
- Askew, M. (2004a). Mediation and interpretation: Exploring the interpersonal and the intrapersonal in primary mathematics lessons. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-78). Bergen: PME.
- Askew, M. (2004b). Objectives driven lessons in primary schools: Cart before the horse? *Proceedings of the British Society for Research into Learning Mathematics*, 24(1), 61-68.
- Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy*. London: King's College.
- Askew, M., & Millett, A. (in press). Critical incidents: When teacher subject knowledge matters. In M. Brown, M. Askew & A. Millett (Eds.), *Learning about number: Interactions and outcomes in primary classrooms*. Netherlands: Kluwer Academic Publishers.
- Askew, M., & Millett, A., & Brown, M. (in press). Changing teaching in Year 4: The impact of a national intervention. In M. Brown, M. Askew & A. Millett (Eds.), *Learning about number: Interactions and outcomes in primary classrooms*. Netherlands: Kluwer Academic Publishers.
- Balfanz, R., Maclver, D., & Brynes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high-poverty middle schools: A multi-site, multi-year study. *Journal for Research in Mathematics Education*, 37(1), 33-64.
- Ball, D. (1993). Halves, Pieces, and Twoths: Constructing and using representational context in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-196). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Ball, D. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle, T. L. Good, & I. F. Goodson (Eds.), *International handbook of teachers and teaching* (pp. 769-818). Dordrecht: Kluwer Academic Publishers.
- Ball, D. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241-247.
- Ball, L., & Stacey, K. (2005). Teaching strategies for developing judicious technology use. In W. Masalski and P. Elliot (Eds.), *Technology-supported mathematics learning environments* (pp. 3-15). Reston: National Council of Teachers of Mathematics.
- Barwell, R. (2005). Working on arithmetic word problems when English is an additional language. *British Education Research Journal*, 31(3), 329-348.
- Baxter, J., Woodward, J., & Olson, D. (2001). Effects of reform-based mathematics instruction on low achievers in five third-grade classrooms. *The Elementary School Journal*, 101(5), 529-549.
- Behrend, J. (2003). Learning-disabled students make sense of mathematics. *Teaching Children Mathematics*, 9(5), 269-273.
- Bicknell, B. (2003). Young children's understanding of volume. *SET: Research Information for Teachers*, 3, 37-39.
- Bicknell, B., & Riley, T. (2005). Addressing mathematical promise. *SET: Research Information for Teachers*, 1, 38-42.
- Biddulph, F. (1996). Children's questions in mathematics: An initial exploration. *SAMEpapers: Science and Mathematics Education Papers*, 74-92.
- Bills, L., & Husbands, C. (2005). Values education in the mathematics classroom: Subject values, educational values and one teacher's articulation of her practice. *Cambridge Journal of Education*, 35(1), 7-18.

- Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36(5), 412-446.
- Bremigan, E. (2005). An analysis of diagram modification and construction in students' solution to applied calculus problems. *Journal for Research in Mathematics Education*, 36(3).
- Brenner, M. (1998). Meaning and money. *Educational Studies in Mathematics*, 36, 123-155.
- Brizuela, B. (2004). *Mathematical development in young children: Exploring notations*. New York: Teachers College Press.
- Brown, J., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-42.
- Burton, L. (1996). Mathematics and its learning as narrative: A literacy for the twenty-first century. In D. Baker & C. Fox (Eds.), *Challenging ways of knowing English, mathematics and science* (pp. 49-55). Reston: NCTM.
- Cahnmann, M., & Remillard, J. (2002). What counts and how: Mathematics teaching in culturally, linguistically, and socio economically diverse urban settings. *The Urban Review*, 34(3), 179-204.
- Calder, N. (2005). "I type what I think and try it": Children's initial approaches to investigation through spreadsheets. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 185-192). Sydney: MERGA.
- Carpenter, T., Fennema, E. & Franke, M. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3-20.
- Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth: Heinemann.
- Carpenter, T., Franke, M., Jacobs, V., Fennema, E., & Empson, S. (1997). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, 29(1), 3-20.
- Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.
- Chatfield, C. (1998). *Problem solving: A statisticians guide*. London: Chapman and Hall.
- Chick, H., Pfannkuch, M., & Watson, J. (2005). Transnumerative thinking: Finding and telling stories within data. *Curriculum matters*, 1, 86-107.
- Christensen, I. (2004). *An evaluation of Te Poutama Tau 2003*. Wellington: Ministry of Education
- Clarke, D. (2001). Understanding, assessing, and developing young children's mathematical thinking: Research as a powerful tool for professional growth. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, pp. 9-26). Sydney: MERGA.
- Clarke, D. (2005). Essential complementarities: Arguing for an integrative approach to research in mathematics classrooms. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 3-17). Sydney: MERGA.
- Clarkson, P. (1992). Language and mathematics: A comparison of bilingual and monolingual students of mathematics. *Educational Studies in Mathematics*, 23, 417-430.
- Cobb, P. (2002). Modeling, symbolizing, and tool use in statistical data analysis. In K. Gravemeijer, R. Lehrer, B. Oers, & V. Leiven (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 171-195). Dordrecht: Kluwer Academic Publishers.
- Cobb, P., & McClain, K. (2001). An approach for supporting teachers' learning in social context. In F. Lin & T. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 207-231). Utrecht: Kluwer Academic Publishers.
- Cooper, B., & Dunne, M. (2000). *Assessing children's mathematical knowledge: Social class, sex and problem-solving*. Buckingham: Open University Press.
- Crooks, T. (1998). The impact of classroom evaluation practices on students. *Review of Educational Research*, 58(4), 438-481.
- Cuoco, A., & Curio, F. (2001). *The role of representations in school mathematics*. Reston, VA: NCTM.
- Davies, N., & Walker, K. (2005). Learning to notice: One aspect of teachers' content knowledge in the numeracy classroom. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematic Education Research Group of Australasia, pp. 273-280). Sydney: MERGA.
- Davis, B. (1997). Listening for differences: an evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, 28(3), 355-376.
- Davis, G., & Hyun, E. (2005). A study of kindergarten children's spatial representation in a mapping project. *Mathematics Education Research Journal*, 17(1), 73-100.
- Davis, R. (1992). Understanding "understanding." *Journal of Mathematical Behavior*, 11, 225-241.
- Davis, R. (1998). The interplay of algebra, geometry, and logic. *Journal of Mathematical Behavior*, 7, 9-28.

- De Geest, E., Watson, A., & Prestage, S. (2003). Thinking in ordinary lessons: What happened when nine teachers believed their failing students could think mathematically. In N. A. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 301-308). Hawaii: PME.
- Diezmann, C. (2002). Enhancing students problem solving through diagram use. *Australian Primary Mathematics Classroom*, 7(3), 4-8.
- Diezmann, C., & English, L. (2001a). Developing young children's multi-digit number sense. *Reoper Review*, 24(4), 11-13.
- Diezmann, C., & English, L. (2001b). Promoting the use of diagrams as tools for thinking. In A. Cuoco (Ed.), *2001 National Council of Teachers of Mathematics Yearbook: The role of representation in school mathematics* (pp. 77-89). Reston: NCTM.
- Diezmann, C., Faragher, R., Lowrie, T., Bicknell, B., & Putt, I. (2004). Exceptional students in mathematics. In B. Perry, G. Anthony & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 175-195). Flaxton: QLD: Post Pressed.
- Diezmann, C., & Watters, J. (1997). Bright but bored: Optimising the environment for gifted children. *Australian Journal of Early Childhood*, 22(4), 14-19.
- Diezmann, C., & Watters, J. (2002). The importance of challenging tasks for mathematically gifted students. *Gifted and Talented International*, 17(2), 76-84.
- Diezmann, C., & Watters, J. (2004). *Challenges and connectedness in the mathematics classroom: Using lateral strategies with gifted elementary students*. Paper presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark.
- Doerr, H., & Lesh, R. (2003). Designing research on teachers' knowledge development. In L. Bragg et al. (Eds.), *MERINO: Mathematics education research: innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 262-269). Sydney: MERGA.
- Doyle, W. (1983). Academic work. *Review of Educational Research*, 53(2), 159-199.
- Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, 23(2), 167-180.
- Draper, J. & Siebert, D. (2004). Different goals, similar practices: Making sense of the mathematics and literacy instruction in a standards-based mathematics classroom. *American Educational Research Journal*, 41(4), 927-962.
- Ell, F., & Irwin, K. (2006, Feb). *Sustained numeracy project practices in two schools*. Paper presented at the National Numeracy Facilitators Conference, Auckland.
- English, L. (2002). Priority themes and issues in international research in mathematics education. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 3-16). Mahwah, NJ: Erlbaum.
- English, L. (2004). Mathematical modelling in the primary school. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 207-214). Sydney: MERGA.
- English, L., & Goldin, G. (2002). The nature of mathematics as viewed from mathematics education research. In A. Cockburn & E. Nardi (Ed.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 141-142). Norwich: University of East Anglia.
- Fennema, E., Carpenter, T., Franke, M., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 27(4), 403-434.
- Forbes, S. (2000). *Measuring students' education outcomes: Sex and ethnic difference in mathematics*. Unpublished doctoral dissertation. Curtin University of Technology, Perth, WA.
- Forgasz, H. J. (2002). Computers for learning mathematics: Equity considerations. In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 260-267). Sydney: MERGA.
- Forman, E., Larreamendy-Joerns, J., Stein, M., & Brown, C. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction*, 8(6), 527-548.
- Forster, P. A., & Mueller, U. (2002). Assessment in calculus in the presence of graphic calculators. *Mathematics Education Research Journal*, 14(1), 16-36.
- Fraivillig, J., Murphy, L., & Fuson, K. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education*, 30(2), 148-170.
- Francisco, J., & Maher, C. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, 24, 361-372.
- Geiger, V. (2003). What students say: Analysis of structured survey data in relation to technology and mathematics learning. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 380-387). Sydney: MERGA.
- Germain, R. (2002). A "positive" approach to supporting a pupil with Down syndrome during 'dedicated numeracy time'. *Down Syndrome Research and Practice*, 8(2), 53-58.
- Gipps, C. (1994). *Beyond testing: Towards a theory of educational assessment*. London: The Falmer Press.

- Goldenberg, P. (1996). Habits of mind as an organizer for the curriculum. *Journal of Education*, 178, 13-34.
- Goos, M. (1999). Understanding mathematical text through peer explanations. In J.M. Truran & K.M. Truran (Eds.), *Making the difference* (Proceedings of the 22 annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 238-245). Adelaide: MERGA.
- Goos, M., & Cretchley, P. (2004). Computers, multimedia, and the Internet in mathematics education. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia* (pp. 151-174). Flaxton: Post Pressed.
- Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2000). Reshaping teacher and student roles in technology-enriched classrooms. *Mathematics Education Research Journal*, 12, 303-320.
- Graham, A., & Thomas, M. (2001). Building a versatile understanding of algebraic variables with a graphic calculator. *Educational Studies in Mathematics*, 41(3), 265-282.
- Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht: CD-B Press.
- Gravemeijer, K. (1997a). Instructional design for reform in mathematics education. In M. Beishuizen, K. Gravemeijer, & E. van Lieshout (Eds.), *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 13-34). Utrecht: Technipress.
- Gravemeijer, K. (1997b). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 315-333). London: Psychology Press.
- Grouws, D., Cooney, T., & Jones, D. (Eds.) (1988). *Perspectives on research on effective mathematics teaching*, vol. 1. Reston, VA: Lawrence Erlbaum Associates for NCTM.
- Groves, S., & Doig, B. (2002). Developing conceptual understanding: The role of the task in communities of mathematical inquiry. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 25-32). Norwich: PME.
- Groves, S., & Doig, B. (2004). Progressive discourse in mathematics classes - the task of the teacher. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 485-502). Bergen, Norway: PME.
- Grugnetti, L & Jaquet, F. (1996). Senior secondary school practices. In A. Bishop (Ed.), *International handbook of mathematics education* (p. 615-646). Dordrecht: Kluwer.
- Guberman, S. (2004). A comparative study of children's out-of-school activities and arithmetical achievements. *Journal for Research in Mathematics Education*, 35(2), 117-150.
- Hatano, G., & Inagaki, K. (1998). Cultural contexts of schooling revisited: a review of the learning gap from a cultural psychology perspective. In S. Paris & H. Wellman (Eds.), *Global prospects for education: Development, culture, and schooling* (pp. 79-104). Washington, DC: American Psychological Association.
- Heid, K. (2005). Technology in mathematics education: Tapping into visions of the future. In W. Masalski & P. Elliot (Eds.), *Technology-supported mathematics learning environments* (pp. 345-366). Reston: National Council of Teachers of Mathematics.
- Helme, S., & Clarke, D. (2001). Identifying cognitive engagement in the mathematics classroom. *Mathematics Education Research Journal*, 13(2), 133-153.
- Henningsen, M., & Stein, M. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524-549.
- Herbert, S., & Peirce, R. (2005). Potential of technology and a familiar context to enhance students' concept of rate of change. In P. Clarkson, A. Downton, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Research, theory and practices*. (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 435-442). Sydney: MERGA.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Human, P., Murray, H. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, May, 12-21.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Higgins, J. (1997). Constraints on peer explanations in junior mathematics classrooms. *SAMEpapers: Science and Mathematics Education Papers*, 128-145.
- Higgins, J. (1998). *Learning and teaching mathematics in the first two years at school: Group work, independence and understanding*. Unpublished doctoral dissertation, Victoria University of Wellington, Wellington, NZ.
- Higgins, J. (2001). *Developing numeracy: Understanding place value* (Report to the Ministry of Education). Wellington.
- Higgins, J. (2003). *An evaluation of the Advanced Numeracy Project 2002*. Wellington: Ministry of Education.
- Higgins, J. (2005). Equipment-in-use in the Numeracy Development Project: Its importance to the introduction of mathematical ideas. In J. Higgins, K. Irwin, G. Thomas, Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 89-96). Wellington: Ministry of Education.
- Higgins, J., Bonne, L., & Fraser, K. (2004). *An evaluation of the Advanced Numeracy Project 2003*. Wellington: Ministry of Education.
- Higgins, J., Irwin, K., Thomas, G., Trinick, T., & Young-Loveridge, J. (2005). *Findings from the New Zealand Numeracy Development Project 2004*. Wellington: Ministry of Education.

- Higgins, J. with Parangi, M., Wilson, R., & Klaracich, Y. (2005). Effective teaching strategies for Māori students in an English-medium numeracy classroom. In J. Higgins, K. Irwin, G. Thomas, Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand numeracy development project 2004* (pp. 74-75). Wellington: Learning Media.
- Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology*, 32(3), 401-415.
- Holton, D., Anderson, J., Thomas, B., & Fletcher, D. (1999). Mathematical problem solving in support of the curriculum. *International Journal of Mathematical Education in Science and Technology*, 30(3), 351-371.
- Holton, D., Spicer, T., Thomas, G., & Young, S. (1996). *The benefits of problem solving in the learning of mathematics* (Research Report). Wellington: Ministry of Education.
- Hong, Y. Y., & Thomas, M. (2001). Conceptual understanding of the Newton-Raphson method. *SAMEpapers: Science and Mathematics Education Papers*, 96-118.
- Houssart, J. (2001). *Setting tasks and setting children*. Paper presented at the British Society of Research into Learning Mathematics.
- Houssart, J. (2005). Count me out: Task refusal in primary mathematics. In A. Watson, J. Houssart & C. Roaf (Eds.), *Supporting mathematical thinking* (pp. 65-73). London: David Fulton Publishers.
- Hunter, R. (2002). *Constructing decimal concepts in an inquiry classroom*. Unpublished Master of Educational Studies, Massey University, Palmerston North.
- Hunter, R., & Anthony, G. (2003). Percentages: A foundation for supporting students' understanding of decimals. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics Education Research: Innovation, Networking, Opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 452-460). Sydney: MERGA.
- Irwin, K. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for Research in Mathematics Education*, 32(4), 399-420.
- Irwin, K., & Britt, M. (2004). Operating with decimal fractions as a part-whole concept. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010*. (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 312-319). Sydney: MERGA.
- Irwin, K., & Britt, M. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Number Project. *Educational Studies in Mathematics*, 58, 169-188.
- Irwin, K., & Niederer, K. (2003). *An evaluation of the Numeracy Exploratory Study Years 7-10 (2002)*. Wellington: Ministry of Education.
- Jacobson, C., & Lehrer, R. (2000). Teacher appropriation and student learning of geometry through design. *Journal for Research in Mathematics Education*, 31(1), 71-88.
- Kazemi, E., & Franke, M. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, 7(3), 203-235.
- Klymchuk, S. (2005). *Counter-examples in calculus*. Auckland: Maths Press.
- Koedinger, K & Nathan, M. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. *The Journal of Learning Sciences*, 13(2), 129-164.
- Lachance, A., & Confrey, J. (2002). Helping students build a path of understanding from ratio and proportion to decimal notation. *Journal of Mathematical Behavior*, 20, 503-526.
- Lampert, M. (1989). Choosing and using mathematical tools in classroom discourse. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 1, pp. 223-346). Greenwich, CT: JAI Press.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27, 29-63.
- Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.
- Lawrence, A., & Patterson, M. (2005). Mathematics word problems and Year 12 students. *SET: Research Information for Teachers*, 1, 34-38.
- Lehrer, R., Schauble, L., Carpenter, S., & Penner, D. E. (2000). The inter-related development of inscriptions and conceptual understanding. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspective on discourse, tools, and instructional design* (pp. 325-360). Mahwah, NJ: Lawrence Erlbaum.
- Lerman, S. (1993). The role of the teacher in children's learning of mathematics. In A.J. Bishop, K. Hart, S. Lerman, & T. Nunes (Eds.), *Significant influences on children's learning of mathematics* (pp. 61-99). Science and Technology Education Document Series No. 47. Paris: UNESCO.
- Lesh, R., & Doerr, H. (2003). *Beyond constructivism: Models and modeling perspectives on mathematical problem solving, learning and teaching*. Mahwah: Lawrence Erlbaum Associates.
- Loretz, M. (2002). *Changes in mathematics assessment tasks and practice as the NCEA is introduced*. Unpublished master's thesis, University of Auckland, Auckland, NZ.

- Lowrie, T. (2004). Authentic artifacts: Influencing practice and supporting problem solving in the mathematics classroom. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 351-358). Sydney: MERGA.
- Lubienski, S. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, 31(4), 454-482.
- Mack, N.K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85-106). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Marton, F., Runesson, U., & Tsui, A. (2004). The space for learning. In F. Marton & A. Tsui (Eds.), *Classroom discourse and the space for learning* (pp. 3-40). Mahwah, NJ: Erlbaum.
- Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum Associates.
- Marzano, R. (2003). *What works in schools: Translating research into action*. Alexandria: Association for Supervision and Curriculum Development.
- Marzano, R., Pickering, D., & Pollock, J. (2001). *Classroom instruction that works: Research based strategies for increasing student achievement*. Alexandria: Association for Supervision and Curriculum Development.
- Masingila, J. O. (1993). Learning from mathematics practices in out-of-school situations. *For the Learning of Mathematics*, 13(2), 18-22.
- Mason, J., & Johnston-Wilder, S. (Eds.). (2004). *Fundamental constructs in mathematics education*. London: RoutledgeFalmer.
- McChesney, J. (2004). *Number sense in a secondary school context*. Unpublished doctoral dissertation, University of Waikato, Hamilton, New Zealand.
- McClain, K., & Cobb, P. (1999). An analysis of the teachers' proactive role in supporting the development of symbolizations. In J. Truran & K. Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, pp. 351-358). Adelaide: MERGA.
- McClain, K., Cobb, P., & Gravemeijer, K. (2000). Supporting students' ways of reasoning about data. In M. Burke (Ed.), *Learning mathematics for a new century* (pp. 174-187). Reston: National Council of Teachers of Mathematics.
- McHardy, J. (2006). *Investigating the impact of a technology integrated flexible programme on senior mathematics students' learning*. Unpublished Master of Science thesis, The University of Auckland, Auckland.
- McKinley, E., Stewart, G., & Richards, P. (2004). Māori students in science and mathematics: junior programmes in secondary schools. *Set*, No 3. Wellington: NZCER.
- Meaney, T., & Irwin, K. (2005). *Language used by students in mathematics for quantitative and numerical comparisons* (NEMP Probe study). Dunedin: EARU, University of Otago.
- Meyer, M., Dekker, T., Querelle, N., & Reys, R. (2001). Contexts in mathematics curriculum. *Mathematics Teaching in the Middle School*, 6(9), 522-527.
- Middleton, J., & Spanias, P. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education*, 30(1), 65-88.
- Middleton, M. (2001). Beyond motivation: Middle school students' perceptions of press for understanding in math. *Contemporary Educational Psychology*, 27, 373-391.
- Ministry of Education. (1992). *Mathematics in the New Zealand Curriculum*. Wellington: Learning Media, Ministry of Education.
- Ministry of Education (2004). *The number framework: Teachers' material*. Wellington: Ministry of Education.
- Ministry of Education. (2006). *Numeracy Development Projects*. Wellington: Learning Media. [Book 1: The Number Framework]
- Ministry of Education. (2006). *Numeracy Development Projects*. Wellington: Learning Media. [Book 2: The Diagnostic Interview]
- Morrone, A. S., Harkness, S.S., D'Ambrosio, B., & Caulfield, R. (2004). Patterns of instructional discourse that promote the perception of mastery goals in a social constructivist mathematics course. *Educational Studies in Mathematics*, 56, 19-38.
- Mousley, J., Sullivan, P., & Zevenbergen, R. (2004). Alternative learning trajectories. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia, pp. 374-381). Sydney: MERGA.
- Moyer, P. (2002). Are we having fun yet? *Educational Studies in Mathematics*, 47, 175-197.
- Mulligan, J., Mitchelmore, M., & Prescott, A. (2005). Case studies of children's development of structure in early mathematics: A two-year longitudinal study. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1-8). Melbourne: PME.
- Nason, R., Woodruff, E., & Lesh, R. (2002). Fostering authentic, sustained and progressive mathematical knowledge-building activity in CSCL communities. In B. Barton, K.C. Irwin, M. Pfannkuch, & M.O.J. Thomas (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 504-511). Sydney: MERGA.

- Norton, S., McRobbie, C., & Cooper, T. (2002). Teachers' responses to investigative mathematics syllabus: Their goals and practices. *Mathematics Education Research Journal*, 14(1), 37-59.
- Nunes, T. (1997). Cognitive invariants and cultural variation in mathematical concepts. *International Journal of Behavioural Development*, 15, 432-453.
- Nunes, T., Light, P., & Mason, J. (1993). Tools for thought: The measurement of length and area. *Learning and Instruction*, 3, 39-54.
- Nunes, T., & Moreno, C. (2002). An intervention program for promoting deaf pupils' achievement in mathematics. *Journal of Deaf Studies and Deaf Education*, 7(2), 120-133.
- O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63-103). New York: Cambridge University Press.
- Pape, S., Bell, C., & Yetkin, I. (2003). Developing mathematical thinking and self-regulated learning: A teaching experiment in a seventh-grade mathematics classroom. *Educational Studies in Mathematics*, 53(3), 179-202.
- Peter, S. (1995). *Young children's ideas about money*. Paper presented at the New Zealand Association of Mathematics Teachers Conference. Auckland.
- Peters, S., & Jenks, J. (2000). *Young children's mathematics: A supporting document for the "Making things count" resource*. Wellington: Victoria University of Wellington.
- Pierce, R., Herbert, S., & Giri, J. (2004). CAS: Student engagement requires unambiguous advantages. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group, Vol. 2, pp. 462-469). Sydney: MERGA.
- Pierce, R., & Stacey, K. (2004). A framework for monitoring progress and planning teaching towards effective use of computer algebra systems. *International Journal for Computers and Mathematics Learning*, 9(1), 59-93.
- Pirie, S., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, 12(2), 127-146.
- Presmeg, N. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23, 595-610.
- Rand Mathematics Study Panel (2003). *Mathematical proficiency for all students: Toward a strategic research development program in mathematics education*. RAND Education Institute.
- Rawlins, P. (in progress). *Students' perceptions of the formative potential of NCEA*. Unpublished PhD thesis, Massey University, Palmerston North.
- Remillard, J., & Bryans, M. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, 35(5), 352-388.
- Resnick, L. B. & Singer, J.A. (1993). Protoquantitative origins of ration reasoning. In T. Carpenter, E Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 107-130). Hillsdale: Lawrence Erlbaum Associates.
- Rietveld, C. (2005). Classroom learning experiences of mathematics by new entrant children with Down syndrome. *Journal of Intellectual & Developmental Disability*, 30(3), 127-138.
- Rosebery, A. (2005). "What are we going to do next?": Lesson planning as a resource for teaching. In R. Nemirovsky & A. Rosebery (Eds.), *Everyday matters in science and mathematics: Studies of complex classroom events* (pp. 299-324). Mahwah: Lawrence Erlbaum Associates.
- Rubick, A. J. (2000). *The statistical thinking of twelve Year 7 and 8 students*. Unpublished master's thesis, University of Auckland, Auckland, NZ.
- Runesson, U. (2005). Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, 35(1), 69-87.
- Salomon, G., & Perkins, D. (1998). Individual and social aspects of learning, In P. Pearson & A. Iran-Nejad (Eds.), *Review of Research in Education*, 23, 1-24. Washington, DC: AERA.
- Saxe, G. (1988). Candy selling and math learning. *Educational Researcher*, 12(6), 14-21.
- Schettino, C. (2003). Transition to a problem-solving curriculum. *Mathematics Teacher*, 96(8), 534-537.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.
- Scribner, S. (1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 9-40). Cambridge, MA: Harvard University Press.
- Sfard, A. (2003). Balancing the unbalanceable: The NCTM Standards in light of theories of learning mathematics. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 353-392). Reston: National Council of Mathematics Teachers.
- Sfard, A., & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning. *The Journal of the Learning Sciences*, 11(2&3), 153-161.
- Sharp, J., & Adams, B. (2002). Children's constructions of knowledge for fraction division after solving realistic problems. *The Journal of Educational Research*, 95(6), 333-347.

- Shaughnessy, J. (1997). Missed opportunities in research on the teaching and learning of data and chance. In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia, pp. 6-22). Sydney: MERGA.
- Sherin, M. (2001). Developing a professional vision of classroom events. In T. Wood, B. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 75-93). Hillsdale: Erlbaum.
- Skovsmose, O. (1993). The dialogical nature of reflective knowledge. In S. Restivo, J. P. van Bendegem, & R. Fischer (Eds.), *Math worlds: Philosophical and social studies of mathematics and mathematics education* (pp. 162-181). Albany, NY: State University of New York Press.
- Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior*, 24, 341-350.
- Stein, M. (2001). Teaching and learning mathematics: How instruction can foster the knowing and understanding of number. In J. Brophy (Ed.), *Subject-specific instructional methods and activities* (Vol. 8, pp. 114-144). Amsterdam: JAI.
- Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
- Stein, M., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, 2, 50-80.
- Stein, M., Smith, M., & Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.
- Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht: Kluwer.
- Sullivan, P., Mousley, J., & Zevenbergen, R. (2003). Being explicit about aspects of mathematics pedagogy. In N. A. Pateman, B. Dougherty & J. Zilliox (Eds.), *Proceeding of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 267-273). Honolulu: PME & PMENA.
- Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). Describing elements of mathematics lessons that accommodate diversity in student background. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 257-264). Bergen: PME.
- Sullivan, P., Warren, E., & White, P. (1999). Comparing students' responses to content specific open-ended and closed mathematical tasks. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the International Group for the Psychology of Mathematics education* (Vol. 4, pp. 249-256). Haifa: PME.
- Sullivan, P., Zevenbergen, R., & Mousley, J. (2002). Contexts in Mathematics teaching: Snakes or Ladders. In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics Education in the South Pacific*, (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 649-656). Sydney: MERGA.
- Swanson, C., & Stevenson, L. (2002). Standards-based reform in practice: Evidence on state policy and classroom instruction from the NAEP state assessments. *Educational Evaluation and Policy Analysis*, 24(1), 1-27.
- Tanner, H., & Jones, S. (2002). Assessing children's mathematical thinking in practical modelling situations. *Teaching Mathematics and its Applications*, 21(3), 135-159.
- Taylor, M., & Biddulph, F. (1994). 'Context' in probability learning at the primary school level. *SAMEpapers: Science and Mathematics Education Papers*, 96-111.
- Thomas, G., & Tagg, A. (2004). *An evaluation of the Early Numeracy Project 2003: Exploring issues in mathematics education*. Wellington: Ministry of Education.
- Thomas, G., & Ward, J. (2002). *An evaluation of the Early Numeracy Project 2001*. Wellington: Ministry of Education.
- Thompson, S., & Senk, S. (2001). The effects of curriculum on achievement in second-year algebra: The example of the University of Chicago School Mathematics Project. *Journal for Research in Mathematics Education*, 32(1), 58-84.
- Thornton, C., Langrall, C., & Jones, G. (1997). Mathematics instruction for elementary students with learning disabilities. *Journal of Learning Disabilities*, 30(2), 142-149.
- Trinick, T. (2005). Te Poutama Tau: A case study of two schools. In J. Higgins, K. Irwin, G. Thomas, T. Trinick & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 80-88). Wellington: Ministry of Education.
- Turner, J., & Meyer, D. (2004). A classroom perspective on the principle of moderate challenge in mathematics. *The Journal of Educational Research*, 97(6), 311-319.
- Vale, C. (2003). Gender and attitudes to computer use for junior secondary mathematics. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 680-687). Sydney: MERGA.
- van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics*, 25(2), 2-9.
- van den Heuvel-Panhuizen, M. (Ed.). (2001). *Children learn mathematics*. Utrecht: Freudenthal Institute.
- van Dijk, I., van Oers, B., Terwel, J., & van den Eeden, P. (2003). Strategic learning in primary mathematics education: Effects of an experimental program in modelling. *Educational Research and Evaluation*, 9(2), 161-187.

- van Tassel-Baska, J. (1997). What matters in curriculum for gifted learners: Reflections on theory, research and practice. In N. Colangelo & G. Davis (Eds.), *Handbook of gifted education* (pp. 126-135). Boston, MA: Allyn and Bacon.
- Vincent, J. (2003). Year 8 students' reasoning in a Cabri environment. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 696-703). Sydney: MERGA.
- Walls, F. (2002). *Sociomathematical worlds: The social world of children's mathematical learning in the middle primary years*. Unpublished doctoral dissertation, Victoria University of Wellington, Wellington, NZ.
- Walls, F. (2004). The New Zealand Numeracy Projects: Redefining mathematics for the 21 century? *The New Zealand Mathematics Magazine*, 41(2), 21-43.
- Warner, L. (2003). *Children's notation of number computation*. Unpublished Master's thesis, Massey University, Palmerston North, New Zealand.
- Watson, A. (2000). Mathematics teachers acting as informal assessors: Practices, problems and recommendations. *Educational Studies in Mathematics*, 41, 69-91.
- Watson, A. (2002). Instances of mathematical thinking among low attaining students in an ordinary secondary classroom. *Journal of Mathematical Behavior*, 20, 461-475.
- Watson, A. (2003). Accordance, constraints and attainments in mathematical activity. In *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 23(2), pp. 103-108). Oxford: BSRL.
- Watson, A. (2003). Opportunities to learn mathematics. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 29-40). Sydney: MERGA.
- Watson, A. (2004). Red herrings: Post-14 "best" mathematics teaching and curricula. *British Education Research Journal*, 52(4), 359-376.
- Watson, A., & De Geest, E. (2005). Principled teaching for deep progress: Improving mathematical learning beyond methods and material. *Educational Studies in Mathematics*, 58, 209-234.
- Watson, A., De Geest, E., & Prestage, S. (2003). *Deep progress in mathematics: The Improving Attainment in Mathematics Project*: Universities of Oxford and Birmingham.
- Watson, A., & Mason, J. (1998). *Questions and prompts for mathematical thinking*. Derby: Association of Teachers of Mathematics.
- Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity*. Mahwah: Lawrence Erlbaum Associates.
- Watson, J. (2000). Statistics in context. *Mathematics Teacher*, 93(1), 54-58.
- Watson, J., & Callingham, R. (1997). Data cards: An introduction to higher order processes in data handling. *Teaching Statistics*, 19, 12-16.
- Watters, J., English, L., & Mahoney, S. (2004). *Mathematical modeling in the elementary school*. Paper presented at the American Educational Research Association Annual meeting, San Diego.
- Wiest, L. (2001). The role of fantasy contexts in word problems. *Mathematics Education Research Journal*, 13(2), 74-90.
- Williams, G. (2002). Identifying tasks that promote creative thinking in mathematics: A tool. In B. Barton, K. Irwin & M. Pfannkuch (Eds.), *Mathematics education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 698-705). Sydney: MERGA.
- Wright, B. (1998). An overview of a research-based framework for assessing and teaching early number learning. In C. Kanes, M. Goos, & E. Warren (Eds.), *Teaching mathematics in the new times* (Proceedings of the 21 annual conference of the Mathematics Education Research Group of Australasia, pp. 701-708). Sydney: MERGA.
- Yelland, N. (2005). Curriculum, pedagogies and practice with ICT in the information age. In N. Yelland (Ed.), *Critical issues in early childhood education* (pp. 224-242). Maidenhead: Open University.
- Young-Loveridge, J. (2005). Students views about mathematics learning: A case study of one school involved in Great Expectations Project. In J. Higgins, K. Irwin, T. Trinick, & J. Young-Loveridge (Eds.), *Finding from the New Zealand numeracy development project 2004* (pp. 107-135). Wellington: Ministry of Education.
- Zevenbergen, R. (2001). Open-ended task: The dilemma of openness or ambiguity? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 447-454). Utrecht: PME.
- Zevenbergen, R. (2005). The construction of a mathematical habitus: Implications of ability grouping in the middle years. *Journal of Curriculum Studies*, 37(5), 607-619.
- Zevenbergen, R., Hyde, M., & Power, D. (2001). Language, arithmetic word problems, and deaf students: Linguistic strategies used to solve tasks. *Mathematics Education Research Journal*, 13(3), 204-218.

6. Mathematics Practices outside the Classroom

Quality pedagogical practice is constituted in social practices (Wenger, 1998). The sorts of practices that teachers participate in to enhance students' cognitive capacities are, of course, not confined to the classroom. Students' mathematical identities and capacities are shaped in ongoing ways by interrelationships between the teacher and the family, whānau, wider school, and professional organisations in which the teacher is involved (Alton-Lee, 2003). These relationships that exist beyond the classroom, and the systems and practices associated with them, provide us with another context for exploring how mathematical knowledge and identities are developed.

Teachers' engagement with mathematics is significantly influenced by the opportunities made available to them within the wider school. In this chapter, we look first at how mathematics teachers practise their profession in ways that are responsive to the school leadership, structures, and organisational processes. We explore the role that school-wide relationships, systems, and professional development initiatives play in the core dimensions of effective classroom practice. Secondly, we investigate the ways in which quality teaching is enhanced by factors outside the school. As documented earlier, home and community represent the two major domains in which the development of young children takes place. We investigate collaborative partnerships between schools and various groups of people from the home and community, "work[ing] together to create better programmes and opportunities for students" (Epstein, 1995, p. 701). What will become apparent are the ways in which links forged between the home culture and the school environment can have beneficial effects on student attitude and academic achievement.

The school

Mathematics instructional practices are informed by teachers' active engagement with processes and people. Precisely because teaching takes place within nested systems of people and structures and develops "in situations where the available information is often partial or incomplete and where the consequences of actions are not always immediate" (Doerr & Lesh, 2002, p. 132), institutional settings are fundamental to the way teachers enact pedagogical practice in the classroom. As such, the school operates as a community of practice in shaping how mathematics is taught and learned.

Within this community, teachers are professionals who modify and transform their pedagogical practice in a generative fashion; they accommodate mandated curriculum in relation to the system-level processes at the school (Spillane, 1999). Teachers are *interpreters and adapters* of curriculum (Shulman & Shulman, 2004; Walshaw, 1995) within the constraints and affordances offered by the school. Goos and Jolley (2004) document how one school governed the ways in which mathematics was taught: its school-wide approach was founded on the common objective of enhancing student learning and placed a strong emphasis on mathematical thinking.

A School-wide Mathematics Philosophy

The school promoted pedagogical effectiveness through its consistent classroom philosophy to mathematics across levels, from kindergarten onwards. Specifically, the school emphasised the strategic use of mathematical knowledge. In junior classrooms teachers emphasised mathematical concepts such as place value and patterning through number and word games. They also encouraged students to make and verify conjectures. For example, pre-school teachers introduced the idea of estimation by asking the children to guess the number of small items in a jar and then think about how they could verify their guess. The students were then asked to compare estimates for two jars of different dimensions. Ways of verifying their results included using the objects to create bar graphs,

as well as simply counting. Throughout, the teachers talked about mathematical thinking, drawing on past experience with similar tasks, and choosing among a variety of acceptable ways to solve the problem.

By the time the students reached year 5 they appeared to be comfortable with the idea that mathematical activity entails attention to strategies and reflection on learning. This was evident in the way they were required to set out their workbooks to make their problem solving strategies explicit. Students at the school were expected to write notes in a double page titled 'Mathematical Thinking'. Sub-headings were included and these were expressed as 'Restatement of question in own words', 'Working', 'Reflection (what did I learn?)', and 'Extension (what if?)'. Across a range of problem solving activities and across the school students used this page as a heuristic to articulate and support their thinking. Evidence of this claim is supported by students' written comments such as: "I think my answer is accurate because I picked up a pattern" and "I learnt that when something is right in front of me I don't have to take the hard way."

From Goos and Jolley (2004)

School leadership

Principals who are respectful of the professional expertise and change intentions of the school's mathematics teaching community significantly influence how reform efforts are implemented (Millett & Johnson, 2004). Coburn (2005) has found that principals, through their greater access to policy messages, directly influence teachers' practice. Principals receive the directives and participate in networking events associated with reform efforts, learning about new materials, approaches, and ideas associated with changing policy" (pp. 499–500). In their discussions with teachers and the provisions they make for learning, principals emphasise certain aspects of curriculum while downplaying others, based on their own understandings.

In a study involving mathematics programmes in New Zealand primary schools, Wood (2003) found that in terms of making professional development opportunities available to staff and providing support to teachers engaged in professional development programmes, the support of principals was critical. Principals in the study consistently reported tensions when trying to balance individual teachers' needs for personal growth with whole-school improvement priorities. Time was a constant theme. For the principals and their teachers, lack of it was a barrier to both the planning and the implementation of new programmes. One principal spoke of it in this way: "That whole thing of having enough time for the amount of information that you are expected to digest and then to actually put that information into practice" (p. 107).

From research, we know that the school leadership team is a central engine of school curricular development (Spillane, 2005). The team has also been shown to be an important factor in teachers' implementation of standards-based assessment. Hipkins et al. (2004) provide evidence that when schools set aside school time for professional discussion and course development, the tasks that accompanied and followed the implementation of NCEA were deemed more manageable by the school. The Education Review Office (ERO, 2004) reported that most teachers valued the time allocated for professional development in connection with NCEA and valued the extra professional dialogue with colleagues.

Continuity within the leadership team has been shown to influence teachers' practice. In her work with disenfranchised students in the US, Gutierrez (2004) points out that teachers who teach mainly socially disadvantaged students, with whom they lack shared life experiences, often have to deal with frequent turnover of leadership and students as well as low staff morale and a disinclination to implement curricular changes. Part of a teacher's practice in these schools "will necessarily entail buffering herself and her students from conflicting practices and maintaining energy to sustain a practice that does not fit with the school context" (p. 3). Gutierrez maintains that a school leadership community that actively contributes to the preparation of students for leadership and active and responsible participation in a democratic society has the following characteristics: "a rigorous and common curriculum, an active

commitment to students, a commitment to a collective enterprise, and innovative instructional practices” (p. 5).

In his investigation into primary school leadership, Spillane (2005) found that school leaders talked with similar emphasis across curriculum areas of leadership functions such as teacher development, programme implementation, and monitoring of teaching practice. But, in his in-depth analysis of leadership practice, he found distinct variations in the way in which leadership is arranged and put into practice in different subject areas. His longitudinal investigation into eight high-poverty schools in the Chicago area revealed that fewer school leaders involved themselves in mathematics-related leadership routines, (e.g., meetings for mathematics teams or to discuss grade levels, curriculum and planning) compared with similar meetings and events for literacy. Typically, school leaders devolved their responsibility for mathematics routines to lead mathematics teachers. Whilst school leaders tended to view the school as the primary change agency for language arts, they looked to external programmes to bring about improvement in mathematics. Similarly, school leaders involved in subject-specific advice networks were less likely to involve themselves in mathematics than literacy.

McClain and Cobb (2004) have documented the critical role that the school leadership team plays in middle grade teacher development. For three and a half years, the researchers worked collaboratively in a school with teachers of 12- to 15-year-old students. They interviewed the teachers to determine their professional development participation, their understanding of the district’s mathematics policies, their accountability lines, their perception of high-stakes test scores, their understanding of assistance given to them and the people who influenced their mathematics teaching. The research found that the school leadership community perceived its role to be one of improving teacher capability in mathematics through the provision of teacher support while keeping an eye on students’ achievement results and teachers’ fidelity to the intent of the curriculum.

In particular, the school provided classroom support by making release time available, resourcing personnel, providing equipment and space, and assisting teachers with professional aspects of their role, such as planning, reflection, and assessment. The leadership team changed the lesson observation forms used in the school so that they became part of a teacher-initiated process instead of a tool for top-down assessment. They rearranged schedules so that teachers could collaborate with each other. They also organised regular meetings between school leaders and mathematics lead teachers to bridge gaps in knowledge and understanding.

In another project involving three schools, Cobb and McClain (in press) looked at how the leadership teams supported teachers in their efforts to improve the quality of teaching and learning. The leaders regularly visited classrooms, focusing specifically on how well the lesson objectives on the boards matched mandated curriculum objectives. They also observed students’ level of engagement and behaviour (Cobb, McClain, Lambert, & Dean, 2003). The leadership teams linked classroom curriculum and lesson structure to test scores, maintaining a watchful eye on student achievement in state-mandated tests. Any fall in test results was brought to the attention of the mathematics leaders. But responsibility for analysing reasons and putting in place remedial strategies was delegated to mathematics leaders. The leadership teams viewed fidelity to the curriculum as evidence of quality pedagogical practice. In one principal’s words, “If you teach the curriculum then the test scores will go up. My job is to make sure they teach the curriculum” (Cobb & McClain, in press, p. 18). In the following vignette, Cobb and McClain expand further on the role of the leadership team.

Active School Leadership

The school leaders have a relatively deep understanding of the general intent of current reform proposals in mathematics education. The team’s vision for mathematics teaching and learning was compatible with that of the leaders of mathematics in their schools. Both viewed teaching as complex and demanding, requiring a deep understanding of both mathematical ideas and students’

mathematical thinking processes. Their agendas involved quality pedagogical strategies as well as the forms that student engagement might take to facilitate outcomes. In turn, classroom teachers who were consistently supported by the leadership team were not reluctant to solicit advice and to develop and refine their practice. In the circumstance when a teacher's practice was perceived to be problematic, the teacher and a member of the school leadership team jointly constructed an improvement plan.

The school leadership team's vision of mathematics teaching and learning developed, in part, as a result of their engagement in numerous professional development seminars during the prior 4 years. In this process the school leaders had experienced teaching consistent with the vision articulated in reform documents. Furthermore they had come to see these competencies as crucial to their role as pedagogical leaders in their schools. For example, the school leaders devoted a proportion of their district-wide biweekly meetings to mathematics. In these settings they completed a pedagogical activity from the curriculum to both develop their own mathematical reasoning and to appreciate better the mathematical intent of the curriculum. These experiences supported their belief that fidelity to the curriculum was the primary means of improving student learning as indicated by test scores.

From Cobb and McClain (2006)

The role of the mathematics lead teacher or head of department in establishing productive teaching communities

The immediate professional community has a marked effect on teacher effectiveness and hence on learner outcomes. It is often the lead mathematics teacher who receives reform directives and who is encouraged to participate in networking events associated with reform efforts or learning about new materials, approaches, and ideas associated with changing policy. Lead teachers bring others 'on board' by working closely with professional development providers. They promote reform ideas by purchasing curriculum materials that focus on one teaching and learning approach to the exclusion of others and by integrating those materials into their mathematics planning.

In a landmark study in the UK, researchers (Millett, Brown, & Askew, 2004) demonstrated that lead mathematics teachers were key players in sustaining a major initiative. In particular, the personal resources of the lead mathematics teacher determined how—and indeed if—reform ideas were taken up by mathematics teachers in a school. Crucially, what was done in classrooms could be attributed in no small way to the lead mathematics teacher's degree of enthusiasm and inclination to make the project work. Effective classroom practice was also attributed to the sharing of ideas and resources amongst teachers. Practical support was important; so too was the emotional support teachers received from each other.

Garden, Wagemaker, and Mooney (1987) report that the support of colleagues and the kind of discussion that takes place during mathematics meetings both have an impact on student outcomes. In pointing out the influences on the performance of the 199 year 9 students who formed the New Zealand sample in the Second International Mathematics Study of the International Association for the Evaluation of Educational Achievement (IEA), Garden et al. highlighted the fact that when teachers centred their mathematics meeting agendas around curriculum content or teaching strategies, their students were more likely to record high gains from pre- and post-testing. Schools that focused on organisational and administrative matters in their meetings were less likely to report high student gains.

Cobb and McClain (in press) report that successful mathematics leaders take the time to analyse the performance of their students. Typically, this analysis is carried out in collaboration with the school leadership team (principal and assistant principal) with the primary purpose of monitoring achievement levels at each grade as well as identifying potential weaknesses in

the curriculum and the adequacy of content coverage. The lead mathematics teachers and leadership teams in the study saw student outcomes as an indicator of both student progress and teacher effectiveness in implementing the curriculum. But it was the mathematics lead teacher who played the key role in conveying curriculum intent to the classroom teachers. “In th[eir] pivotal role as brokers between their own and the other communities, the [lead teachers] had at least partial access to the practices of both the professional teaching and the school leadership community” (McClain & Cobb, p. 285).

In the following vignette, Cobb and McClain provide us with further insights into the critical leadership role played by the lead mathematics teacher in creating effective teacher communities. Teachers in their research drew on their colleagues for support as classroom resources. In placing students and their needs at the centre of their work, they developed and shared instructional materials, they regularly communicated and reflected on students and teaching, they sometimes planned their lessons jointly, they took turns at writing course assignments, and they panel-marked assignments. Pedagogical practice at these schools was to some extent influenced by the relationships and the systems within the institution.

An Effective Teachers’ Learning Group

As part of their process of organising for mathematics teaching, three mathematics teacher leaders conducted biweekly meetings with their teachers working at a particular grade level in three different middle-grade schools. Although the mathematics teacher leaders gave priority to the implementation of the curriculum and adherence to the State Standards in these grade-level meetings, their larger goal was to support the teachers’ development of pedagogical practices that would support students’ development of mathematical understanding. To achieve this goal, the mathematics teacher leaders focused on the teachers’ understanding of the mathematical intent of pedagogical activities as they addressed implementation issues. To this end they and the teachers worked together to complete pedagogical activities and examined student work on these and similar activities.

One classroom teacher described this emphasis in the grade-level meetings as follows:

“I would call it a grade-level learning group. It’s a grade-level maths meeting where you go in and you usually pick the topic at the prior meeting [based] on where you’ll be. That’s where you go in and really look at what you are studying, how close your students are getting. We take a bit of time doing that, then we may tear apart the [mathematics] book. We may sit and look at a fraction book, we have two fraction books. We may say, this is really redundant, these are the same lessons—let’s do one and take the other out for expediency’s sake. Or we may say, you know, this is really crucial so we need more lessons. I know people are reading this and that, but look at this lesson over here and what relates to it. And this one and this one. And this one. We may bring in articles that we found were valuable. Or we may say, you know what, I have no idea what the point is, I have no idea what the form or the function is. We can sit and discuss how important that is, and how it works.”

From Cobb and McClain (2006)

Making time available within the school day for teachers to engage in such fruitful discussions is fundamental to teacher community development and hence to student outcomes. Time plays another key role in student achievement: time apportioned by the institution to the teaching and learning of mathematics significantly influences student performance (Garden et al., 1987). The more time there is available for the teaching and learning of mathematics, the higher the achievement of students. The way in which a teacher allocates time to mathematics is also a good predictor of student gains. Reporting on secondary school student achievement in the IEA Study, Garden et al. (1987) found that teachers of high-gain classes spent more time teaching mathematics. While both high- and low-gain classes devoted some of the allocated class time to routine administration, teachers of high-gain classes tended to spend less time than others maintaining order and discipline within the class.

Support from the professional community

A study undertaken by [Steinberg, Empson, and Carpenter \(2004\)](#) found what proved to be a powerful means of accessing student thinking: the presence of a ‘knowledgeable’ mathematics resource person in the classroom who could observe, describe, and unpack critical moments that the classroom teacher had overlooked. The presence of such a resource person also gave the teacher the confidence to try out new ideas and new pedagogical approaches to exploring student thinking. Unfortunately, resource people are not always on hand in the classroom. Another approach that [Kazemi and Franke, 2004](#) and others (e.g., [Ball & Cohen, 1999](#); [Lin, 2002](#)) have found conducive to teachers’ reflecting-in-action is their participation in a professional community. There is a wealth of evidence that shows how reflecting-in-action is developed through the support and encouragement received within a professional community of learners ([Dufour, 2004](#); [Higgins, Irwin, Thomas, Trinick, & Young-Loveridge, 2005](#); [Sherin & Han, 2004](#)). Within such a community, teachers engage in and reflect on mathematical teaching experiences ([Shulman & Shulman, 2004](#)) and “participate in a professional discourse that includes and does not avoid critique” ([Wilson & Berne, 1999, p. 195](#)). The professional learning community might involve researchers and teachers ([Little, 1999](#)) engaged in a collaborative learning endeavour. Equally, it might be defined through the interactions between the teacher and a mathematics resource teacher, numeracy facilitator, or mathematics syndicate or department ([Jaworski, 1994](#))—all of whom can provide the teacher with observational or written material for reflecting on students’ mathematical understandings.

Expanding on the advantages of collaborative pedagogical work, [Little \(1999\)](#) proposes that reflection on teaching practice is most profoundly effective when the professional learning community engages in the “systematic, sustained study of student work, coupled with individual and collective efforts to figure out how that work results from the practices and choices of teaching” (p. 235). [Hammerness, Darling-Hammond, and Bransford \(2005\)](#) highlight the point in the following vignette.

Attending to Students’ Thinking

A second grade teacher asked students to solve $3 + 3$. One boy, whom we’ll call Jimmy, excitedly answered that the answer was 8. After asking him to rethink and still hearing the same answer, the teacher held up three fingers on each hand and asked Jimmy to count them. This time he got the answer “6.” Great,” said the teacher, “so what is $3 + 3$?” Jimmy again said “8,” leaving the teacher perplexed.

Eventually it was discovered that Jimmy was highly visual and considered “8” to be the answer because a 3 and a reversed 3 made 8 visually. Initially it took considerable time for the teacher to understand the reasons for Jimmy’s answer (which was far preferable than simply saying “you are wrong” and not helping him understand why).

Once the teacher understands Jimmy’s reasoning, it should become much easier (more efficient) for her to diagnose similar answers from others who might also have a proclivity to think visually about these kinds of problems. Adding this information to the teachers’ repertoire of familiar (routine) problems helps her become more likely to handle new sets of novel (non-routine) teaching problems that may occur subsequently.

From Hammerness et al. (2005)

[Kazemi and Franke \(2004\)](#) document key shifts that occurred in the course of a year as the result of monthly workgroups. Their research is informed by the precepts of Cognitively Guided Instruction (CGI), a research and professional development programme ([Carpenter & Fennema, 1992](#); [Carpenter, Fennema, Franke, Levi, & Empson, 1999](#)). In their work with teachers, the researchers found that the most powerful change occurred when their research teachers were engaged in investigations into students’ thinking. Franke and colleagues have distinguished

five levels of sophistication in teachers' engagement with students' mathematical thinking. Their hierarchical order identifies degrees of pedagogical aptitude when it comes to negotiating between instructional approaches, teaching principles, and students' contributions:

- Level 1. The teacher does not believe that the students in his or her classroom can solve the problems unless they have been taught how. Does not provide opportunities for solving problems. Does not ask the students how they solved problems. Does not use students' mathematical thinking when making instructional decisions.
- Level 2. The teacher begins to view students as bringing mathematical knowledge to learning situations. Believes that students can solve problems without being explicitly taught a strategy. Talks about the value of a variety of solutions and expands the types of problems they use. Is inconsistent in beliefs and practices related to showing children how to solve problems. Issues other than student's thinking drive the selection of problems and activities.
- Level 3. The teacher believes it is beneficial for children to solve problems in their own ways because their own ways make more sense to them and the teacher wants the students to understand what they are doing. Provides a variety of different problems for students to solve. Provides an opportunity for the students to discuss their solutions. Listens to students talking about their thinking.
- Level 4A. The teacher believes that students' mathematical thinking should determine the evolution of the curriculum and the ways in which the teacher individually interacts with students. Provides opportunities for students to solve problems and elicits their thinking. Describes in detail individual students' mathematical thinking. Uses knowledge of thinking of students as a group to make instructional decisions.
- Level 4B. The teacher creates opportunities to build in students' mathematical thinking. Describes in detail individual students' thinking. Uses what he or she learns about individual students' mathematical thinking to drive instruction.

(Franke et al., 2001, p. 662)

In their research, Fennema and colleagues (1996) found that at the end of a four-year intervention, 19 out of 21 teachers were teaching at level 3 on this scale or higher. At the beginning of the intervention, the researchers had identified nine teachers at level 1 and seven at level 2. Using the same scale, Franke et al. (2001) showed that movement through the levels requires a supportive community who will pose questions about students' thinking for teachers to reflect upon. Later, teachers will initiate this questioning themselves and it becomes part of their own pedagogical practice. Teachers in their research who engaged in reflection-in-action were able to change their teaching in ways that were both sustainable and self-generative, and that enhanced the quality of mathematical interactions.

Whole-school partnerships

Trinick (2005) explored whole-school effects on Māori-medium mathematics teaching. Like other researchers (e.g., Bishop, Berryman, & Richardson, 2001; Hohepa, 1993) investigating Māori-medium education, he noted the key role of culture and the importance of whole-school partnerships. In his investigation into system-wide processes, Trinick studied two schools that had participated in Te Poutama Tau in 2003. Both had achieved positive mean stage gains on the Number Framework and both were highly committed to the project. Findings revealed that lead mathematics teachers played a significant role in the way in which the Numeracy Project was implemented. In 2003, Te Poutama Tau facilitators and teachers reported results from initial and final diagnostic interviews involving 1339 students. The results reveal encouraging gains in student achievement when compared with achievements recorded during the previous year's pilot programme. Teachers' enthusiasm for, and success with, the project

was influenced by their “willingness to change and a desire to improve teaching practice; a workload that allowed adequate time for teachers to focus on the programme; and good classroom management skills” (Christensen, 2004, p. 23). Teachers were less likely to engage with the intent of the reform if school-wide support was not available.

The senior staff in Trinick’s (2005) study agreed that their success with the Te Poutama Tau initiative in their Māori-medium classrooms was also due in part to their schools’ focus on cooperative learning, which aligned well with the teaching and learning philosophy of Te Poutama Tau. In collaboration with staff, leadership teams monitored student performance and developed clear goals and expectations. The two principals concerned provided release time and financial support for classroom and lead teachers as well as additional support for a number of classroom teachers. But a significant factor in the success of the programme was the principals’ hands-on involvement, which initiated a shared sense of purpose amongst staff. This was important, given the complex changes to teaching practice required by the project. Both principals attended professional development and progress meetings with the numeracy facilitators and worked directly alongside teachers. They modelled dispositions, language and actions symptomatic of the programme and both were able to generate enthusiasm and enhance teachers’ belief in their own capabilities. Pedagogical change became a collaborative problem-solving activity for the principals, classroom teachers, and lead teachers at the two schools.

From research, we know that such collaborative processes are not always at work in schools. Kensington-Miller (2004) documents difficulties in implementing professional development strategies in low-decile secondary school classrooms. Ten teachers in four schools worked with researchers involved in The Mathematics Enhancement Project. The researchers report that meetings with teachers were difficult to organise because communication by email regularly went unanswered and phone messages were often not passed on. The four schools operated according to their own individual timetables for sporting events, student examinations, ERO visits, and report writing. These different schedules made it difficult for the researchers to co-ordinate teacher meetings. Organising peer observations also proved difficult, particularly because teachers were reluctant to take up the offer of classroom release time for this purpose. Mentoring teachers was less than straightforward in the case of teachers who were sensitive to the need to justify their pedagogical approach. Although teachers reported that they valued the opportunity to access literature, some found reading an onerous task and none implemented any of the ideas in their own practice.

Balfanz, MacIver, and Byrnes, (2006) also report on the implementation of mathematics reforms in a high-poverty district. In a four-year effort to develop comprehensive and sustainable reforms in three US middle schools, the researchers report that only a moderate level of implementation was attained. Principally, the impact of the project was constrained by high levels of teacher mobility, both within schools (between grades and subjects) and between schools. As the researchers say, successful implementation is “very difficult to achieve and sustain on a broad scale when every year a substantial number of students are being taught by teachers who are new to the curriculum or the grade” (p. 58). They add that when many students in a school are behind grade level, substantial assistance needs to be found to provide effective extra help.

Clarke (2001) finds that effective teacher development hinges on school and district leadership teams providing substantive rather than merely tacit support for mathematics teaching and learning. Specifically, “the support of the school and district administration, students, parents and the broader school community” (p. 22) was a feature that enhanced learning for students in the Australian state of Victoria’s Early Numeracy Research Project (ENRP). The professional teaching community also directly influences teachers’ development. In an evaluation of the impact on teachers of the Count Me In Too early numeracy programme in New South Wales, Bobis (2004) reports that one of the real impediments to successful implementation was the expressed disapproval of the programme by other teachers. Opposition had a more marked

effect on teachers who were in the early stages of their teaching career. Where support was forthcoming, the professional community provided a climate for sharing knowledge and for sharing thinking about what counts as effective instruction.

Walshaw and Anthony (2006) provide evidence of an empowering team of teachers who developed their own subject knowledge as they worked at facilitating their students' social and cognitive development. The teachers were engaged in collaborative action in their implementation of the Numeracy Development Project (NDP). The NDP is part of the Ministry of Education's Numeracy Strategy, which has as its primary objective the raising of student achievement in numeracy through lifting teachers' professional capability. The NDP has been shown to be a particularly effective development model (Alton-Lee, 2006, p. 5). Teachers' enthusiasm for and success with the project is influenced by the support and encouragement they receive from each other and from the leadership teams in their schools. Within the supportive community of colleagues, teachers have engaged in professional discourse that has confronted the hard issues that surround new approaches to teaching.

Walshaw and Anthony interviewed principals, lead mathematics teachers, numeracy teachers and new teachers in 12 schools across the decile rankings. Based on teacher reports, Walshaw and Anthony show that system-level support, as well as collegial debate, discussion, and reflection on pedagogical practice, allows classroom teachers to improve practice in specific ways. For the teachers in their study, this support provided a lens through which they were able to interpret, adapt, and transform the intent of the reforms. Their sense making of the reforms emerged through patterns of social interaction with colleagues and the norms that shaped and structured priorities. The way in which teaching was organised, and the human, material, and financial support offered by the school, significantly influenced pedagogical effectiveness. Conversations with peers, grounded in everyday efforts to enact curriculum ideas and exemplifying a 'norm of collaboration and deliberation' (Spillane, 1999) enabled classroom teachers to grasp what the numeracy reforms meant for classroom practice. The following vignette illustrates one school's effective school-wide approach to teacher practice and student performance.

A School-wide Approach to Numeracy Teaching

The school had made significant commitment to the New Zealand Numeracy Development Project in terms of finance, time and resourcing. School-wide expectations and accountability measures at the decile 3 school, to some extent, pressed classroom teachers to attend carefully to the reform proposals. The principal noted that "twice a year we collect information from school-wide assessment and from that we set our targets. At Year 8 I want them to be at a certain stage. I looked at our data and thought, 'that's not good enough for them to be going off to high school'."

The principal explained how the school had organised extensive support for teachers in his school. Professional support was centred initially on an expert working in isolated classrooms and modelling lessons. Support didn't stop there: the expert "came back to check on teachers. We had a teacher who wasn't fulfilling the obligation and she came and worked alongside that teacher. It wasn't just 'Go off, have a day, and then go back and do it'," he explained. "There was that on-going thing."

Colleagues as well as expert facilitators were central to the school's reform efforts. Still centred on the classroom, a school-wide system was developed whereby individual teachers chose one senior teacher in the school to "come in and observe a specific aspect of the mathematics programme that the teachers decided on." Feedback was provided immediately afterwards. As the principal said, "that way we're actually continuing the professional development." Peer observation not only allowed teachers to determine if practice was consistent with the numeracy programme intent and to sort out pedagogical or content problems, it also provided teachers with the motivation to improve their practice.

The school's lead mathematics teacher explained that they took a whole-school approach to mathematics teaching: "We share across the school the different things that we are doing. And so we

do things like that to help our planning and to help our organisation.” Collegial support and feedback on practice, as well as the sharing of individual classroom efforts, created incentives for teachers to think seriously about and formalise their ideas about effective practice. From their team meeting deliberations the teachers had produced a document that synthesised their collective ideas about effective numeracy classrooms. The schedule established for them the characteristics of effective teaching and the features of an empowering classroom environment for their particular students and for their particular school.

Teachers at the school were able to exert a degree of control over their professional development. Their involvement in the core professional business of the school gave them a sense of affiliation and mission. And the support and encouragement they gave to and received from each other was recognition that they were all valued as professionals.

From Walshaw and Anthony (2006)

School–home partnerships

Effective and sustainable relationships between the home, community, and school are based on the premise of shared responsibilities and mutual investment in students’ well-being. The Program for International Student Assessment (PISA) demonstrated that parental involvement affects students’ academic attainment in no small measure. “When parents interact and communicate well with their children they can offer encouragement, demonstrate their interest in their children’s progress, and generally convey their concern for how their children are faring, both in and out of school” (OECD, 2004, p. 166). Research has highlighted the importance of providing families, irrespective of socio-economic background, with information about and support for their participation as ‘active partners’ in the production of their children’s education (Alton-Lee, 2003; McNamara et al., 2000; **Sheldon & Epstein, 2005**).

Biddulph et al. (2003) make particular mention of the Johns Hopkins University Centre on School, Family and Community Partnerships and its groundbreaking research on the development of effective collaboration between these groups (e.g., Epstein, Simon, & Salinas, 1997; Henderson & Berla, 1994). Epstein (1992) has argued that “students at all levels do better academic work and have more positive school attitudes, higher aspirations and other positive behaviours if they have parents who are aware, knowledgeable, encouraging and involved” (p. 1141). Research conducted by Epstein (1987) demonstrates that teachers who collaborate with parents come to understand their students better and negotiate meanings with parents and students. Parents benefit too: they develop a greater understanding of the school’s programme and an appreciation of their children’s understandings (McBride, 1991).

As the principal agents of mathematics education, teachers have an obligation to work with parents and community to develop an understanding of the relevance for future, informed citizenship of mathematics at school and in the home. Conclusive research has shown that parents tend to value teaching in the school and in the home if it makes a significant contribution to the social and emotional as well as cognitive aspects of their children’s development (de Abreu & Cline, 2005). Building school–home/community partnerships is at the heart of the New Zealand Literacy and Numeracy Strategy (Ministry of Education, 2001) and parent and community involvement in the teaching of mathematics has been an important factor in the successful implementation of the Numeracy Projects in many schools.

Parents’ beliefs and expectations for their children concerning mathematics, and parent–teen mathematical discussions have both been associated with student achievement in elementary, middle and high school mathematics (Entwisle & Alexander, 1996; Gill & Reynolds, 1999; Ho & Willms, 1996). Wang and Lin (2005) explored this association via a systematic analysis of comparative research focusing on Chinese and other specific groups of students within the US. They wanted to tease out the factors and social conditions that contribute to the well-documented mathematical success of students from China (see OECD, 2004; Hiebert, 2003).

The study highlighted the importance of family values and processes and of parents' high expectations of and support for their children's mathematical education.

Like that of Ran (2001, cited in Biddulph et al., 2003), the study undertaken by Wang and Lin found that Chinese families actively engage in informal and formal instructional approaches with their children in the home. Huntsinger et al. (2000) provide first-hand evidence of this practice. They tracked matched groups of 40 Chinese American and 40 Caucasian American students and their parents from preschool to fourth grade. On the basis of yearly test scores, parent interviews and observations of parent-child interactions, Huntsinger and colleagues report that the Caucasian students were outperformed by the Chinese American students, whose parents had allocated more time to formal and systematic instruction in the home. This is not to suggest that home instruction *caused* the higher gains but to note that home involvement, along with many other factors and conditions, does play a part in mathematical success.

There are, of course, many good reasons why parents might not be involved in, or might even show resistance to, their children's mathematics education. Parental hesitancy to participate is sometimes influenced by their own unhappy mathematical experiences and lack of confidence in their ability to help their child (Bryan, Burstein, & Bryan 2001). Gal and Stoudt (1995) have found that parents frequently do not have the requisite content knowledge, that they may not be familiar with the culture of the mathematics programme or the school's teaching approaches, and that they may not receive guidance from teachers about the ways in which they can help. Parents know what it means to *read* with children, yet they are often unclear about what it means to *do mathematics*. OECD (2004) has also signalled that single parents coping with the dual responsibilities of work and children's education often lack the time and resources to help with homework, volunteer in the classroom, or attend meetings with teachers or principals.

Sukon and Jawahir (2005) carried out a survey involving 1800 fourth grade students in 60 primary schools and found that parental education, availability of reading material in the home, family possessions, and parental support for education in the home all influence students' numeracy achievement. Results from their survey, conducted in Mauritius, showed that home-related factors explained 24.6% of variation in numeracy achievement. Parental income has also been associated with student success. OECD (2004) noted that primary-aged children whose parents have a low level of education and low incomes or are working in low-prestige occupations or are unemployed are unlikely to achieve as well as children living in socio-economically advantaged situations.

This finding needs to be set alongside evidence from a study undertaken by Lubienski (2002) in the US. This research recorded no differences in terms of support from parents of diverse SES. Parents of low SES are just as keen as other parents to encourage and support their children in their mathematics education. Similarly, a New Zealand study (Walshaw, 2006) found that, irrespective of school decile ranking, parents were seriously concerned that their children should develop the mathematical skills and know-how that would equip them to take advantage of post-school opportunities. In their BES of community and family influences on student achievement, Biddulph et al. (2003) report on McKinley's (2000) study that showed that Māori families "wanted their children to do well at school ... [and that] their children required an education to get a job and hoped that their efforts would be enough to achieve that" (p. 118). Biddulph et al. note, too, that "many Tongan parents were prepared to make considerable sacrifices to enhance their children's educational achievement because they believed in socio-economic mobility through education" (p. 118). The same was true for the parents of the secondary school Pasifika girls in Jones' (1986) study. Parents in that study viewed education as a means to access a wider range of post-school options than had been available to the parents themselves.

Pritchard (2004) surveyed the parents of a small, inner-city, New Zealand primary school and found they were positive about helping their children with mathematics and had many ideas on

how they could assist their children at home. These ideas included shopping, cooking, measuring, counting, and making comparisons. The parents also suggested discussing strategies, revisiting mathematics problems, playing board games, and making links with music. In addition, they offered practical examples for developing conceptual understandings for fractions, patterns, and relationships and assistance with modelling and rewriting mathematical questions. Tizard and Hughes (1984), as well as Savell and Anthony (2000), claim that home cultures that have a close affinity to the school culture tend to contribute to higher student achievement. Savell (2000) noted that, for families where children were experiencing mathematical difficulties, school newsletters offering information and mathematical activities for families were seen as a ‘test’ or an exercise, while the families of high achievers saw them as an opportunity for incorporating mathematics into their shared daily lives.

This is an important finding, given that parental attitudes and perceptions of mathematics have been found to influence not only student learning outcomes (Hall & Davis, 1999; Horne, 1998), but also the development of student self-efficacy (Lehrer & Shumow, 1997; Tiedemann, 2000). Savell and Anthony (2000) note: “It is possible that rather than levelling the playing field (so that high- and low-contact parents can all be informed about classroom programmes) the mathematics newsletters may have a differential effect: they may be advantaging some children and not others, thereby widening the gap between high- and low-achieving children” (p. 53).

The issue of differential effect is, however, complex and it concerns, among other things, the issue of motivation. D’Amato (1992) has provided conclusive evidence of differential motivation amongst students. D’Amato defines motivation as a relation between students and the practice established in the classroom rather than an inherent, stable trait of students. Students’ development of mathematical interests reflects one of two ways identified by D’Amato for how learning mathematics in school can have value to students. D’Amato refers to this source of value as *situational* significance, where students come to view engagement in mathematical activities as a means of gaining experiences of mastery and accomplishment, and a means of maintaining valued relationships with peers and teachers. D’Amato contrasts situational significance with what he terms *structural* significance. In this case, students come to view achievement in mathematics as a means of attaining other ends, such as entry to college and high-status careers or acceptance and approval in household and other social networks.

It has been well documented that not all students have access to a structural rationale for learning mathematics in school. Gutiérrez (2004) observes, for example, that many urban students do not see themselves going to college. They may hold activist stances, have more pressing daily concerns (e.g., housing, safety, healthcare), or may not believe that hard work and effort will be rewarded by future educational and economic opportunities. D’Amato (1992), Erickson (1992), and Mehan, Hubbard, and Villanueva (1994) all document that students’ access to a structural rationale varies as a consequence of family history, race, or ethnic history, and class and caste structures within society. Failure of schools and classroom teachers to recognise diversity and to provide all students with access to a situational rationale for learning mathematics creates a barrier to the formation of fruitful mathematical identities.

Facilitating school, family, and community partnerships

There is evidence that parental conversations with mathematics educators can encourage parents to be more positive towards mathematics and towards investigating their children’s mathematical thinking (Civil, 2002). Goldring (1991) goes as far as to say that some teachers spend more time and energy on children of high-contact parents. Sanchez and Baquedano (1993) showed that students whose parents met with teachers and were informed about ways they might help at home registered higher mathematical achievement than those whose parents did not receive information. Parents who attended training and information workshops about how to help their preschool (Starkey & Klein, 2000) or primary school children at home (Shaver & Walls, 1998) appeared to contribute more significantly to their children’s mathematics performance than did the parents who did not attend such sessions.

Sheldon and Epstein (2005) described the kinds of home practices that promote students' mathematical understanding. The following vignette illustrates how effectively implemented activities mobilise family involvement and contribute to students' success at school.

Home Activities that Contribute to Mathematical Understanding

Sheldon and Epstein explored the efforts of schools to develop relationships between family, community and school, and the effect of those involvement activities on student performance in mathematics. Drawing on longitudinal data from 18 primary and secondary schools with largely economically disadvantaged student populations, these researchers found that, after controlling for previous levels of mathematics achievement, successful implementation of targeted supporting practices in the home correlated positively to greater student mathematics gains on standardised tests. Mathematics-focused learning-at-home activities were consistently associated with improved student performance.

School, family and community partnerships were forged for 18 schools in an initiative to improve students' mathematical understanding. All of the schools reported that they (a) provided parents with information on how to contact mathematics teachers, (b) scheduled meetings with parents of students who were struggling with mathematics and (c) reported to parents on student progress and problems in mathematics. Those common practices were rated among the most effective for helping students improve their mathematics achievement.

Schools used a range of activities for varied purposes, and the activities for each type of involvement varied in effectiveness. For example, evening workshops for parents were rated more effective than daytime workshops. Teacher-designed interactive homework and mathematics materials for families and students to use at home were rated more positively for boosting students' skills than were videotapes. Certificates were issued to students to recognise mastery of specific mathematics skills by fewer schools than those who employed the more traditional lines of communication. However, the strategy was rated highly by the schools that used it.

School leaders expressed high levels of confidence that family and community involvement activities can help improve student learning and achievement in mathematics. Activities that supported mathematics learning included homework assignments that required students and parents to interact and talk about mathematics. Mathematics materials and resources made available to families to use at home also supported mathematics learning. The relationships between implementation of these activities and mathematics achievement were strong and positive, even after accounting for the influential variables of schools' prior achievement or level of schooling. The reported quality of implementation, rather than mere use of material or activity, was strongly and consistently associated with changes in levels of student mathematics achievement.

From Sheldon and Epstein (2005)

Sheldon and Epstein urge schools to plan strategically for family-involvement activities that will encourage and enable curriculum-relevant interactions between students and family members. In what ways might schools make it possible for parents to participate actively in their children's mathematics education? Epstein (1995) proposes a typology of six categories of involvement and support:

- Parenting: Helping all families establish supportive home environments for children.
- Communicating: Establishing two-way exchanges about school programmes and children's progress.
- Volunteering: Recruiting and organising parent help at school, home, or other locations.
- Learning at home: Providing information and ideas to families about how to help students with homework and other curriculum-related materials.
- Decision making: Having parents from all backgrounds serve as representatives and

leaders on school committees.

- Collaborating with the community: Identifying and integrating resources and services from the community to strengthen school programmes.

Communication

Walshaw (2006) found that, where specific school and community initiatives are lacking, New Zealand parents are less able to fulfil an effective supporting role. Peters (1998) reported that parents of new entrants felt uninformed and would welcome information about the school and about how their children were dealing with a new environment, the skills they were developing, and the content they were learning. They preferred teachers to schedule time with them for formal meetings rather than meet with them on a casual basis. Similarly, the six parents interviewed by Evers and Young-Loveridge (2005) expressed their frustration at a lack of information about what their children were doing at school and how they, as parents, could help. “Initially, all the parents ... believed that their children were doing well at mathematics, but on reflection ... three of them felt that they had no evidence on which to base this assumption” (p. 46). “Two parents had ‘absolutely no idea’ about what their children were achieving” (p. 45), and “five out of the six parents said that they knew ‘nothing’ or ‘not a lot’” (p. 45) about the mathematics their children were doing in class.

Two-way sustainable relationships and open communication pathways are necessary if the content and teaching and learning approaches of the classroom are to be familiar to families. Mutual relationships also ensure that the needs and experiences of the home community are made relevant in the classroom. In a study undertaken by Carey (1998), communication was found to be a key lever for encouraging parents to engage with their children’s mathematical development. Carey found that the parents in one multi-ethnic classroom were keen to assist in any way possible, but many considered themselves lacking the skills to engage in parent–student dialogue about mathematics. The teacher in this urban school recognised that these parents needed greater fluency with English, so began by assisting them to improve their general English language skills and specialist mathematics vocabulary. Only then did she ask them to help their children with mathematical problems. She kept up the communication with parents, informing them of the particular skills their children were working on.

Parents want to be informed of their children’s classroom experiences. Parent–teacher communication through newsletters was listed as the best first step by parents in a study undertaken by Cattermole and Robinson (1985). Parents in that research more readily assisted in mathematical activities when they received information about the kinds of mathematical activities the class were involved in. Pamphlets that describe the Numeracy Projects and give some simple yet effective ways in which parents can encourage and support their children’s mathematics learning (Ministry of Education, 2004, 2005) have been particularly useful for many parents. Written materials such as these help parents become familiar with the pedagogical practices in their child’s classroom, as do hui or meetings with the classroom teacher and hands-on experience with the mathematics equipment being used (Evers & Young-Loveridge, 2005).

Mothers in the study by Savell (1998), however, reported that teachers avoided them rather than communicated with them: “It is made quite clear that you are not allowed to waltz in and out of the classroom ... [The teacher] doesn’t even want you sitting on the seat outside waiting. She says that if the child sees you, that distracts them and she doesn’t want that happening” (p. 530). Peressini (1998) has underlined the issue of parent alienation from school mathematics activities and shown how school practices that deter parents from active participation contrast markedly with the rhetoric of partnership found in educational initiatives. Teachers and governance structures that conceptualise the home as subservient to the school when it comes to mathematics development make it difficult for parents to be involved, except as spectators, in their child’s education. Hornby (2000) goes as far as saying that some teachers have negative attitudes towards parents, viewing them more as problems or adversaries than collaborative partners.

Salway and Winter (2003), in their Home School Knowledge Exchange Project, undertaken in the UK, attempted to draw on the rich experiences, skills, and knowledge of the communities of four schools. A home-school book was established, in which students filed weekly information sheets describing the mathematics currently being taught and outlining related activities that parents and children could collaborate on in the home. In some schools, parents participated in class with their children, and some parents attended mathematics workshops. Board games played at home were shared with others in the classroom. Some schools utilised student profiles as a means of finding out about students' out-of-school interests and activities.

In the following vignette, Goos and Jolley (2004) document a school-generated partnership between teachers, parents, and students across year levels. In this school, it is the principal who, with a long-time interest in community partnerships, is instrumental in forming productive working partnerships across the community. She is supported in her goals by like-minded and enthusiastic people in the community.

Creating Productive Partnerships between Teachers, Parents and Students

An urban middle-class school worked at creating productive partnerships between teachers, parents and students. The school's perspective on parental involvement emphasised communicating and learning at home through strategies such as the inclusion of a "Maths Corner" in the school newsletter and the provision of individualised "take-home packs" of mathematics activities to parents who requested additional materials to use with their children. However, the most enriching initiatives for some parents were the fortnightly Maths for Parents sessions, where the Principal and teachers discussed topics identified by parents (usually involving current curriculum issues and pedagogical approaches) and participation in a mock literacy and numeracy test after which the Principal worked through the paper and discussed the kind of mathematical knowledge and skills being assessed.

At these sessions parents were concerned they were not familiar with current mathematics teaching approaches that differed from their own experiences at school. For example, some parents expressed anxiety that their children were not learning 'tables' by rote. There was a clear feeling from the parents that children should be drilled (and by implication tested) on 'tables' and given more homework. This was despite the fact that they remembered hating learning tables and said that they themselves never mastered mathematics. Teachers spent some time demonstrating to parents that there are many efficient strategies for mental and written computations. They emphasised that developing this kind of flexibility and fluency was encouraged in today's schools. They also offered suggestions for ways in which parents might meaningfully incorporate mathematical thinking into everyday activities such as sharing out food or comparing the shapes of traffic signs.

From Goos and Jolley (2004)

Decision making and collaboration: curriculum development and activities

A small number of international research projects highlight the ways in which parents and the wider community have participated in the development of mathematics curriculum (e.g., Lipka, 1994). Meaney (2001) provides evidence of a New Zealand community's involvement in curriculum development. She documents how she facilitated a Māori school as they negotiated a curriculum for their kura kaupapa Māori. Parents at the school were expected to take an active role in decision-making processes. The research was premised on the understanding that collaboration between parents and teachers over mathematics content and teaching approaches in the kura would build productive links between the home and school cultures. Sharing curriculum decision making with the school community went a significant way towards creating sustainable, meaningful collaboration.

The framework that Meaney designed served as a catalyst for deliberations on the mathematics curriculum that were "based on, but not exclusively, Māori preferred teaching and learning

methods” (Smith, 1990, p. 148). Sufficiently open to allow for modification, the framework consisted of two parts: process and issues. The process aspect included discussion, reflection, research, and decision making by consensus. The issues aspect involved mathematics in the school and the community, teaching and learning mathematics, the sequence of student learning, the language of instruction, assessment, teacher professional development, and resources.

The Māori community discussed and debated how mathematics should best be taught to their children. Using the framework as a starting point, most of the dilemmas raised were relevant to the community’s own particular circumstances and conditions. Since these dilemmas were context-driven, they did not necessarily have application for other communities. In the main, the discussions were focused not so much on the production of a new curriculum as on purposeful engagement with others and sharing understandings and beliefs with a view to ensuring that the curriculum would be relevant and meaningful for the community. As a consequence of the discussions, two teachers assumed responsibility for organising ongoing meetings with the aim of producing a mathematics curriculum policy document.

Civil (in press) reports on the work she has carried out on curriculum development for working-class, mostly Latino communities. Central to her ongoing work is a belief that community knowledge and experience are strategic resources for schooling. Civil has gathered evidence that students contribute actively to household functioning by interpreting for parents and other family members, helping care for younger siblings, assisting in the economic development of the household (through, for example, assisting with the repair of cars and appliances), and by participating in cultural customs and traditional ceremonies. Because students’ participation and learning is often at odds with their experiences at school, the teacher of a year 4 and 5 class developed a module designed to allow students to experience classroom learning that was more closely aligned to their out-of-school learning. In the following vignette, Civil records a successful implementation that preserves the purity of the students’ funds of knowledge while maintaining fidelity to the mathematics.

The Garden Module

A garden module was developed by the researchers and teachers after interviews with families revealed that parents held considerable knowledge in relation to gardening. The module became an opportunity for students to build on their gardening experiences while they were engaged in mathematically rich tasks (e.g., exploring how area varies given a fixed perimeter and discussing different ways to graph and scale the growth of an *Amaryllis*).

The teacher was well-liked and respected by the parents. She viewed the parents as intellectual resources and drew them in, in ways that were very different from typical parental involvement, allowing them to become co-constructors of the curriculum. The families contributed actual resources (such as seeds and soil) as well as offering personal expertise with gardening. The teacher learned from the parents and families and had some of them come in as experts. She engaged with them in conversations and these conversations helped her shape the curriculum.

In one activity, each group of 4 or 5 students made a garden enclosure using a 3 foot long string that each student glued to paper in their chosen shape. The challenge for them was to find the area of that shape, using tools (if they so chose) such as cubes, tiles and rulers to help them. The different shapes and their areas were displayed and the students discussed what shape would yield the largest area. Despite the artificiality of the activity the students were intrigued and curious about the problems of how to find the area of an irregular shape and how to maximise the area while keeping the perimeter constant.

Their experience with the in-class activity made its way into their gardens as many of the groups started working towards making their garden circular—although some acknowledged that in bigger gardens a circular design could be problematic in terms of access to plants.

From Civil (in press)

A large number of studies undertaken in a range of countries have demonstrated that parent involvement in classroom instructional tasks [see also Early Years chapter] leads to positive learning outcomes (e.g., Campbell & Mandel, 1990; Coleman, 1998; Rosenholtz, 1989; Sanders & Epstein, 1998). Fullan (1991) provides evidence that parent involvement (in the capacity of either a volunteer or an assistant) has an impact on instruction over and above all other forms of parental involvement. What these studies have found is that student learning benefits from parent involvement in the classroom when parents are given the opportunity to act as ‘subteachers’, providing instruction that models, reinforces, and supports the teacher’s instructional purposes. Such involvement also helps develop student attitudes and behaviours that are typically associated with productive school performance.

Kyriakides (2005) documents the successful implementation of a partnership policy in Cyprus that encouraged parents to work with their children in the classroom while teaching was taking place. The project involved parents and teachers as collaborators, with parents being assigned the role of advisor, learner, and teacher aide rather than the role of ‘house-keeper.’ The teachers drew on the life experiences of the parents and used them as instructional resources. The parents’ classroom activities were incorporated into the planning and every effort was made to clarify for the parents the intent of the lesson. Time was scheduled for parent feedback. When compared with the control group six months later, the students in the experimental school showed greater improvement in attainment in mathematics and in other subjects. The partnership fostered positive attitudes among parents and students. Parents reported that home–school links had been consolidated and that student behaviour in the home had improved.

Parent–teacher collaboration was a key factor in the success of a mathematics programme in a Māori-medium mathematics classroom. Trinick (2005) presents a compelling example of two-way communication between family/whānau and the school.

Kura A Implements a Māori-medium Mathematics Programme

Kura A is a rural full primary decile 2 school of 245 students, 94% are Māori. The school has a long history of productive home–school relationships, of strong bonds with the local community and of an identification with a vibrant iwi. The school has an even distribution of Māori-medium and English-medium classes. Many of the parents attended the school as students and expressed a desire for their children to be immersed in the reo and traditions of their own iwi. In some cases, children live with grandparents who still reside on the local area so that they can go to this particular school.

The school actively promoted community involvement in Te Poutama Tau, in order to develop a shared sense of purpose and direction. The school reported regularly to families on students’ progress and recorded significant changes in teachers’ and students’ attitudes to pangarau. A large number of whānau attended the hui held to introduce Te Poutama Tau and were fully supportive of the initiative in the school. In this process the Te Poutama Tau facilitators played a significant role. Parents/caregivers were invited to participate in the development of aspects of the teaching and learning programme, particularly on te reo and tikanga. The principal and other senior staff visited the local contributing kōhanga to provide information to staff and parents about classroom routines and offer an introduction to the Number Framework. The principal felt that this strategy assisted in preparing kōhanga graduates for entry into primary school.

From Trinick (2005)

Homework

One way in which “a culture of parents as partners within the school community” (Merttens, 1999, p. 80) can be created is through parent involvement in home activities. Indeed, often the only link between home and school is through formal homework (Anthony & Knight, 1999). Despite the fact that many parents recognise regular homework more for its character-

building potential than for its educational value (Merttens, 1999), generations of parents have been involved in sustained, collaborative partnerships with schools through their support for homework (Peressini, 1998). For example, parents have monitored and assisted their children as they have worked on mathematics homework tasks assigned by teachers for out-of-classroom completion. New Zealand year 9 students work on mathematics homework for up to one hour per day (Garden, 1996), and students aged 15 years spend slightly more time on English, mathematics, and science homework combined than do many other students (OECD, 2003).

Over a half of all students in the IEA Study (Garden, Wagemaker, & Mooney, 1987) received help with mathematics from other family members. Students in the year 9 study who demonstrated high gains between pre- and post-testing were less likely to receive regular assistance than students who achieved low gains. It has been found that as students move into secondary school and deal with more advanced mathematical content, parents are less likely to assist with homework. Meredith (2005) surveyed New Zealand secondary school students for their views on homework and found that class friends were the primary source of assistance for year 11 and 13 students. Only 13% of year 13 parents were in a position to help with homework activities. In contrast, year 9 students reported that they received help with homework—mostly from their mothers.

De Abreu and Cline (2005) report that new immigrant families in their UK study tend to rely on siblings to help with homework: “we don’t have much time for the children with regards maths ... because it’s an English medium here ... They won’t understand, so therefore the brothers and sisters help more at home. If there’s a deficiency in them, that they don’t understand then they obtain help from their brothers and sisters rather than us” (p. 714). Homework centres have proven to be another form of successful community support. Tongan families in New Zealand benefited from such an initiative. Biddulph et al. (2003) report on Fusitu’a and Coxon’s (1998) study of a successful venture set up to assist the Tongan secondary school students of one high school with their homework.

Research has found that homework that precipitates family interactions opens up communication lines between home and school. It ensures parents’ participation in their children’s mathematical schooling and helps raise student performance. However, time spent on homework is not a reliable indicator of student achievement. “How [homework] is done is more important than that it is done, because the how will make the difference between supporting children’s learning and facilitating the collaboration of their parents, or it becoming yet another element in an education system in which the benefits are differentially available, according to socio-economic class, gender or ethnicity” (Merttens, 1999, p. 79).

Homework that is purposeful and engaging, rather than homework that demands sustained periods of time, lends itself to family interactions and discussion (Crystal & Stevenson, 1991). Families that encourage and support mathematics at home are likely to advance students’ performance on mathematics at school. Sheldon and Epstein (2005) conducted research in primary, middle, and secondary schools. Students from the 18 research schools recorded higher proficiency levels when the take-home mathematics activities encouraged parent-child interactions. After accounting for prior levels of mathematics proficiency in the school, the researchers found that mathematics-focused, learning-at-home activities that required students to talk about and interact with their families were consistently and positively related to improvements in mathematical proficiency as measured by achievement tests.

Merttens (1999) and Peressini (1998) provide clear evidence that mathematics homework that relates directly to current school topics and that invites productive dialogue between parent and child plays a key role in students’ cognitive and affective development. Parents’ own experiences and knowledge act as catalysts in developing students’ understanding, particularly in problem-solving activities in realistic contexts (Ford, Follmer, & Litz, 1998). Involvement like this, made possible by the school, enables the parent to become an insider, “a participant in a discipline, rather than someone viewing the discipline entirely from the outside” (Bereiter, 1994, p. 22).

Home activities

A number of researchers have found that mathematical activities made available in the home are conducive to students' cognitive and affective development (e.g., Bragg, 2003; Cutler et al., 2003; Ernest, 1997). As we noted in chapter 4, pre-schoolers engage from an early age and in varying degrees (Young-Loveridge, 1989) with their parents in number-based activities and conversations when interacting with puzzles, toys, television, computer programs, and other games (Tudge & Doucet, 2004). Particular activities have been shown to enhance cognitive development. For example, Biddulph et al. (2003) document evidence that reveals a positive association between card and board game playing and early mathematical competence.

Street, Baker and Tomlin (2005) report a mother's comment on numeracy development as a consequence of her year 2 daughter's involvement in a game of Monopoly.

She suddenly knows \$8, five, a one and a two, or two threes and a two to make eight, because she had to pay you \$8. She soon worked it out, three and three is six and two is eight. It's very small numbers but it's amazing how quickly a lot of children her age can't cope with going round the board and the money. (p. 69)

The daughter played a variety of games, including Monopoly, Snap, and a shopping game, at home with her mother, younger sister, and older cousins. The mother had previously told researchers of her own concern about helping her daughter with mathematics: "My help with maths is going to be very limited ... I've already told my husband that's his department, maths" (p. 64). However the mother (Trisha) was able to use Monopoly to help her daughter (Anne) develop confidence with counting.

Trisha: Can you make 12 with 3s, Anne?

Anne: Oh yeh, I can. Well these 3s go like you've got 3, and 3 makes 6, and another 3 will make 9, and another 3 makes 12. With that it makes 15.

Trisha: [to researcher] I didn't know she could do 3s.

Trisha: [to Anne] Can you count in 5s?

Anne: [quickly] 5, 10, 15, 20, 25.

Trisha: Did you do that in school?

Anne: No, you taught me!

(Street, Baker, and Tomlin 2005, p. 69)

Money

Home activities and practices come into play in students' mathematical development. One particular home activity may assist in enhancing classroom learning. Abranovitch, Freedman and Pliner (1991) found that six- to ten-year-old Canadian children who were given pocket money or allowances seemed more sophisticated about money than those who were not. Students who received pocket money in the form of a weekly or monthly allowance—given either unconditionally or for some household work—revealed a more developed understanding about saving, planning, and keeping to money plans. Berti and Monaci (1998) implemented a 20-hour instructional session on banking with 25 third grade middle-class students over a two-month period. While students in a control group showed very limited understanding, most of those in the experimental group revealed a grasp of banking fundamentals. The researchers developed the programme around the teaching of new arithmetic skills and suggest that banking offers a relevant and meaningful context for number work.

In her study involving Hawaiian students from early years to year 2, Brenner (1998) explored the money activities engaged in at home, out shopping, and in classroom lessons. The investigation showed that students constructed different knowledge at different levels. More importantly, differences between school and everyday money experiences led the researcher to question

the purpose of everyday mathematics in the curriculum. Brenner says: “the goal of bringing mathematics into the classroom is not to recreate the everyday experience” (p. 153). Rather, she argues, everyday topics should be used in the classroom to harness and reconcile the powerful mathematical knowledge students learn at school.

Different home activities are practised within different cultures. Guberman (2006) investigated the association between students’ out-of-school activities and arithmetical achievements and the role that ethnicity plays in these. The study revealed that year 1–3 Latin American students engage in instrumental activities with money, while Korean students tend to engage in money activities that are expected to support their learning. The students’ achievements on arithmetic tasks matched their out-of-school activities: Latin American students were more successful with tasks that involved money while Korean American students were more successful with tasks that involved the use of denominational chips. Guberman suggests that “a culturally relevant pedagogy ... is likely a more appropriate approach, especially for teachers working in multicultural settings” (p. 145).

Games and books

Games and books provided at centre or school have been shown to enhance numeracy levels (Young-Loveridge, 2004). Young-Loveridge’s initiative examined the effectiveness of commercially published books and games on new entrants’ numeracy development.

New Entrants Using Mathematical Books and Games

In a New Zealand intervention based on tutoring pairs of students, 151 new entrants from low decile schools used number stories, rhymes and games to enhance numeracy development. Students were initially directly involved either as intervention or control students. 44% were Māori, 8% were from Pasifika or other cultures, and 48% were Pākehā. The immediate and long-term benefits of using commercial publications of books and games were closely examined for 23 students, and compared to 83 non-users. The intervention session took place each weekday for 30 minutes over a seven-week period. The intent of the programme was to develop children’s numeracy skills across ethnic groupings. In particular the programme focused on developing children’s knowledge of number word sequences, on improving their accuracy, reliability and automaticity in using the enumeration process, on enhancing their experience with forming collections of particular sizes, and on developing their knowledge of stylised (spatial) number patterns and numerals.

The games library was extensive and included commercial games such as Snakes and Ladders, Bingo, Make a Set, Tiddly Winks, and dice with dot patterns or numerals. Levels of difficulty were adjusted to ensure appropriate challenge for each child. The extensive books library included *The Very Hungry Caterpillar*, *A Weta has Six Legs*, and *Ten Silly Sheep*. The session was introduced with a number rhyme and was followed by a number story. The children were encouraged to check the quantities described in the story and predict the next quantity. A familiar board game, chosen by the children, was played and followed by participation in an unfamiliar game. The games ordinarily involved reading the numeral or recognising the number pattern displayed in the dice, and counting out the movement of counters in the game. In some games children counted small wooden sticks or the numerals on playing cards. A contrast group of students used the resources of their classroom BSM programme rather than the intervention programme games and books.

Children in the intervention group, using real authentic story books and games, made substantially greater gains than the contrast group over the intervention period. Effect sizes were as high as 1.99 immediately after the programme ceased, and most students sustained the knowledge 15 months after the intervention. Significant gains were recorded for students’ knowledge of number sequence, stylised number patterns, numeral identification, and forming and adding small collections. The specialist teacher was fully occupied in managing the session and her guidance and support was critical to the success of the intervention. Indeed a key feature of the programme was the role that this teacher played in supporting and structuring levels of challenge to advance students’ learning.

From Young-Loveridge (2004)

The support and guidance provided by the specialist teacher in Young-Loveridge's study might also be provided by parents. Of course, the instructional value of authentic games and books found in the home or sent home from school with students is dependent on parents' availability and willingness to provide guidance and support. It is also crucially dependent on parents' understanding of the mathematical purpose of the activity, the role they play, and the ways in which the activity builds on classroom work (Peters, 1994). Parents need to know the rationales for activities and approaches undertaken at school. De Abreu and Cline (2005) report on parents' reluctance to encourage specific classroom activities in the home. As one parent in the multi-ethnic study said, "When we were young, in school, we didn't use these calculator things ... if you learn by heart, you always remember" (p. 709).

Jennings et al. (1992) found that children's mathematics story books provide an effective means to teach mathematics. Working with five-year-olds from predominantly single-parent families, the researchers reported a statistically significant improvement in students' numerical understanding. Peters (1994) examined the effectiveness of a mathematics initiative with year 3 children and their parents. Once a week for 15 weeks, parents participated in classroom mathematical games with small groups of children. Peters reports on the difficulty experienced in getting parents to engage; those who did participate actively often lacked the skills to enhance student learning. These findings suggest that teachers should be made aware of the critical importance of providing skills development and support for parents. An initiative that has focused on developing parents' skills so that they can work with their children is the Family Math program. The EQUALS programmes at Berkeley University, California, established Family Math to help parents support their children's learning by engaging together in mathematical activities.

Hughes and Greenhough (1998) observed five- to seven-year-olds playing one version of a mathematical game with their teacher and another version with a parent at home. The game was based on one used in the IMPACT project (Merttens & Vass, 1993). The two versions of the game, the Snail Game and the Train Game, required simple addition skills as well as simple ideas about chance. All children played the two versions: half commenced with the Snail Game and the other half commenced with the Train Game. Most children made connections between the two activities: some were concerned with game-playing procedures and others with appropriate strategies. What the study revealed was that similar contextual and content factors inherent in an activity will provide students with sufficient information to make connections between home and school environments. As Hughes and Greenhough note, the study lends support to the classroom practice of sending home mathematics activities, with the proviso that the activities have sufficient similarity in both locations.

Ell (1998) reports on one school's approach, which allowed parents to become directly involved with their children's mathematics education. Parents of five-year-olds and their teachers at this primary school established a games library, which acted as a bridge between learning at home and learning at school. In this investigation, one group of parents joined in the games, taking turns along with the children and talking about the progress of the game in the way that a young player would. A second group of parents took on more of an instructional role, scaffolding learning and introducing more advanced concepts. Ell reports that the nature of the interactions and the extent of engagement during game playing were both critical to the level of student learning that took place.

Conclusion

In exploring pedagogical practices and their impact on student performance, the literature has highlighted key aspects that lie outside the classroom walls. Taking Wenger's (1998) characterisation of community as our basis for effective practice, we have investigated factors and conditions that complement and enhance classroom practices. We have paid close attention to the wider school, home, and community and the role that these play in students' academic and social outcomes. These relationships, and the systems and practices associated with

them, have provided us with another context for exploring how mathematical knowledge and identities are shaped.

What this exploration has revealed is that quality teaching is a joint enterprise involving mutual relationships and system-level processes that are shared by school personnel. Quality teaching in mathematics is a resource rather than blueprint, adapted by teachers within the dynamics of the spaces they share with other professionals in their schools. Research has provided clear evidence that effective pedagogy is founded on the material, systems, human and emotional support, and resourcing provided by school leaders as well as the collaborative efforts of teachers to make a difference for all learners. Teacher relationships and capability, in turn, contribute to student performance (McClain & Cobb, 2004).

Empirical evidence also reveals the beneficial effects of collaborative and sustainable relationships with teachers, family, and whānau. The home, the school, and the community comprise the major domains in which students live, learn, and grow. It is in these domains that shared interests in, and responsibilities for, children are recognised. The relationships and processes that cross over within these spheres contribute in no small way to enhanced student achievement. They are fundamental to the creation of classroom environments that affirm the identities, experiences, aspirations, and knowledge of students. Cross-overs that are fruitful involve partnerships; these involve and require two-way communication, sustained mutual collaboration, encouragement, and support in effort and activities. Initiatives by schools that result in cultures of productive partnership have the effect of creating greater opportunities for students; they become important sources for students' cognitive and affective growth.

References

- Abramovitch, R., Freedman, J. L., & Pliner, P. (1991). Children and money: Getting an allowance, credit versus cash, and knowledge of pricing. *Journal of Economic Psychology*, 12, 27-45.
- Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis*. Wellington: Ministry of Education.
- Alton-Lee, A. (2006). *Iterative best evidence synthesis: Strengthening research, policy and practice links to improve outcomes*. Paper presented at the 4th annual Policy Conference, New Zealand: Policy Evolution.
- Anthony, G., & Knight, G. (1999). The role of practice in mathematics learning. *The Mathematics Educator*, 4(2), 91-107.
- Balfanz, R., Maclver, D., & Brynes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high-poverty middle schools: A multi-site, multi-year study. *Journal for Research in Mathematics Education*, 37(1), 33-64.
- Ball, D.L., & Cohen, D.K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession* (pp. 3-31). San Francisco: Jossey-Bass.
- Bereiter, C. (1994). Constructivism, socioculturalism and Popper's world 3. *Educational Researcher*, 23(7), 21-23.
- Berti, A. E., & Monaci, M. G. (1998). Third graders' acquisition of knowledge of banking: Restructuring or accretion? *British Journal of Educational Psychology*, 68, 357-371.
- Biddulph, F., Biddulph, J., & Biddulph, C. (2003). *The complexity of community and family influences on children's achievement in New Zealand: Best evidence synthesis*. Wellington: Ministry of Education.
- Bishop, R., Berryman, M., & Richardson, C. (2001). *Te toi huarewa. Report to the Ministry of Education*. Wellington: Ministry of Education.
- Bobis, J. (2004). Time, resources, information overload and classroom management: Issues surrounding professional development. In A. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 103-110). Sydney: MERGA.
- Bragg, L. A. (2003). Children's perspectives in mathematics and game playing. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 160-167). Sydney: MERGA.
- Brenner, M.E. (1998). Meaning and money. *Educational Studies in Mathematics*, 36, 123-155.
- Bryan, T., Burstein, K., & Bryan, J. (2001). Students with learning disabilities: Homework problems and promising practices. *Educational Psychologist*, 36(3).

- Campbell, J., & Mandel, F. (1990). Connecting math achievement to parent influence. *Contemporary Educational Psychology, 15*, 64-74.
- Carey, L. (1998). Parents as math partners: A successful urban story. *Teaching Children mathematics, 4*, 314-319.
- Carpenter, T.P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. In W. Secada (Ed.), *Researching educational reform: The case of school mathematics in the United States* (pp. 457-470). Special Issue of the *International Journal of Educational Research*.
- Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L., & Empson, S.B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.
- Cattermole, J., & Robinson, N. (1985). Effective home/school communication: From the parents' perspective. *Phi Delta Kappan, 67*(1), 48-50.
- Civil, M. (2002). Culture and mathematics: a community approach. *Journal of Intercultural Studies, 23*(2), 133-148.
- Civil, M. (in press). Building on community knowledge: An avenue to equity in mathematics education. In N. S. Nasir & P. Cobb (Eds.), *Diversity, equity, and access to mathematics education*. New York: Teachers College Press.
- Clarke, D. (2001). Understanding, assessing, and developing young children's mathematical thinking: Research as a powerful tool for professional growth. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 9-26). Sydney: MERGA.
- Cobb, P., & McClain, K. (2006). The collective mediation of a high stakes accountability program: Communities and networks of practice. *Mind, Culture, and Activity, 13*, 80-100.
- Cobb, P., McClain, K., Lamberg, T., & Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and district. *Educational Researcher, 32*(6), 13-24.
- Coburn, C. (2005). Shaping teacher sensemaking: School leaders and enactment of reading policy. *Educational Policy, 19*(2), 476-509.
- Coleman, P. (1998). *Parent, student and teacher collaboration: The power of three*. London: Sage.
- Crystal, D. S., & Stevenson, H. W. (1991). Mothers' perceptions of children's problems with mathematics: A cross-national comparison. *Journal of Educational Psychology, 83*(3), 372-376.
- Cutler, K. M., Gilkerson, D., Parrott, S., & Bowne, M. T. (2003). Developing math games based on children's literature. *Young Children, 58*(1), 22-27.
- D'Amato, J. (1992). Resistance and compliance in minority classrooms. In E. Jacob & C. Jordan (Eds.), *Minority education: Anthropological perspectives* (pp. 181-207). Norwood, NJ: Ablex.
- de Abreu, G., & Cline, T. (2005). Parents' representations of their children's mathematics learning in multiethnic primary schools. *British Educational Research Journal, 31*(6), 697-722.
- Doerr, H., & Lesh, R. (2002). A modelling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A model and modelling perspective on mathematics teaching, learning and problem solving* (pp. 125-139). Mahwah, NJ: Lawrence Erlbaum Associates.
- Dufour, R. (2004). What is a professional learning community? *Educational Leadership, 61*(8), 7-11.
- Education Review Office. (2004). Progress of NCEA Levels 1 and 2, and readiness of schools to implement Level 3. Wellington. <http://www.ero.govt.nz>
- Ell, F. (1998). *School-initiated mathematics in the home: Five-year-olds and their parents interact*. Unpublished master's thesis, University of Auckland, Auckland, New Zealand.
- Entwisle, D.R., & Alexander, K.L. (1996). Family type and children's growth in reading and math over the primary grades. *Journal of Marriage and Family, 58*, 341-355.
- Epstein, J. L. (1987). Parent involvement: What research says to administrators. *Education and Urban Society, 19*(2), 119-137.
- Epstein, J. L. (1992). School and family partnerships. In M. Akin (Ed.), *Encyclopedia of educational research* (2nd ed., pp. 1139-1151). New York: Macmillan.
- Epstein, J. L. (1995). School/family/community partnerships: Caring for the children we share. *Phi Delta Kappan, 76*, 701-712.
- Epstein, J. L., Simon, B. S., & Salinas, K. C. (1997). *Involving parents in homework in the middle grades* (Research Bulletin No. 18). Bloomington, IN: Phi Delta Kappa/CEDR.
- Erikson, F. (1992). Transformation and school success: The policies and culture of educational achievement. In E. Jacob & C. Jordan (Eds.), *Minority education: Anthropological perspectives* (pp. 27-51). Norwood, NJ: Ablex.
- Ernest, P. (1997). Popularisation: Myths, massmedia and modernism. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education: Part 2* (pp. 785-817). Dordrecht: Kluwer Academic Publishers.
- Eyers, G., & Young-Loveridge, J. (2005). Home-school partnerships in mathematics education. *SET: Information for Teachers, 1*, 43-47.
- Family Mathematics: www.lhs.berkeley.edu/equals/FMnetwork.htm

- Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 27, 403-434.
- Ford, M. S., Follmer, R., & Litz, K. K. (1998). School-family partnerships: Parents, children and teachers benefit! *Teaching Children Mathematics*, 4(6), 310-312.
- Franke, M.L., Carpenter, T.P., Levi, L., Jacobs, V.R., & Empson, S.B. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, 38, 653-689.
- Fullan, M. (1991). *The new meaning of educational change*. London: Cassell.
- Fusitu'a, L., & Coxon, E. (1998). Ko e 'Ulungaanga Faka-Tonga mo e Ako Lelei: Tongan culture and academic achievement. *New Zealand Journal of Educational Studies*, 33(1), 23-38.
- Gal, I., & Stoudt, A. (1995). *Family achievement in mathematics*. NCAL Connections. Philadelphia: National Center on Adult Literacy, University of Pennsylvania.
- Garden, R., Wagemaker, H., & Mooeny, C. (1987). Explaining mathematics achievement. In R. Garden & D. Carpenter (Eds.), *Mathematics achievement in New Zealand secondary schools. A report on the conduct in New Zealand of the Second International Mathematics Study within the International Association for the Evaluation of Educational Achievement*. Wellington: Department of Education.
- Gill, S., & Reynolds, A.J. (1999). Educational expectations and school achievement of urban African children. *Journal of School Psychology*, 37, 403-424.
- Goldring, E. B. (1991). Parents: Participants in an organisational framework. *International Journal of Educational Research*, 15(2), 215-227.
- Goos, M., & Jolly, L. (2004). Building critical partnerships with families and communities to support children's numeracy learning. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 279-286). Sydney: MERGA.
- Guberman, S.R. (2006). A comparative study of children's out-of-school activities and arithmetical achievements. *Journal for Research in Mathematics Education*, 35(2), 117-150.
- Gutierrez, R. (2004). The complex nature of practice for urban (mathematics) teachers. Paper presented at the Rockefeller Foundation Conference, *Investigating the practice of school improvement: Theory, methodology, and relevance*. Bellagio, Italy.
- Hall, C., & Davis, M. (1999). Gender and racial differences in mathematical performance. *Journal of Social Psychology*, 139(6).
- Hammerness, K., Darling-Hammond, L., Bransford, J. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world* (pp. 358-389). San Francisco: Jossey-Bass.
- Henderson, A. T., & Berla, N. (Eds.) (1997). *A new generation of evidence: The family is critical to student achievement*. Columbia, MD: National Committee for Citizens in Education.
- Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., Chui, A., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. Washington, DC: National Centre for Education Statistics.
- Higgins, J., Irwin, K.C., Thomas, G., Trinick, T., & Young-Loveridge, J. (2005). *Findings from the New Zealand Numeracy Development Project 2004*. Wellington: Ministry of Education.
- Hipkins, R., Vaughan, K., Beals, F., & Ferral, H. (2004). *Learning curves: Meeting student learning needs in an evolving qualifications regime. Shared pathways and multiple tracks*. http://www.nzcer.org.nz/default/php?products_id=665.
- Ho, E. S., & Willms, J.S. (1996). Effects of parental involvement on eighth-grade achievement. *Sociology of Education*, 69, 126-141.
- Hohepa, M. (1993). *Preferred pedagogies and language interactions in te kōhanga reo*. Auckland: Research Unit for Māori Education, University of Auckland.
- Hornby, G. (2000). *Improving parental involvement*. London: Cassell.
- Horne, M. (1998). Linking parents and school mathematics. In N. Ellerton (Ed.), *Issues in mathematics education: A contemporary perspective* (pp. 115-135). Perth: MASTEC.
- Hughes, M., & Greenhough, P. (1998). Moving between communities of practice: Children linking mathematical activities at home and school. In A. Watson (Ed.), *Situated cognition and the learning of mathematics* (pp. 127-141). Oxford: Oxford University Press.
- Huntsinger, C. S., Jose, P. E., Larson, S. L., Krieg, D.B., & Shaligram, C. (2000). Mathematics, vocabulary, and reading development in Chinese American and European American children over the primary school years. *Journal of Educational Psychology*, 92(4), 745-760.
- Jennings, C. M., Jennings, J. E., Richey, J., & Dixon-Krauss, L. D. (1992). Increasing interest and achievement in mathematics through children's literature. *Early Childhood Research Quarterly*, 7, 263-276.
- Jones, A. (1986). "At school I've got a chance..." Social reproduction in a New Zealand secondary school. Unpublished doctoral dissertation, Auckland University.
- Kazemi, E., & Franke, M.L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry.

- Journal of Mathematics Teacher Education*, 7, 203-235.
- Kensington-Miller, B. (2004). Professional development of mathematics teachers in low socio-economic secondary schools in New Zealand. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 320-327). Sydney: MERGA.
- Kyriakides, L. (2005). Evaluating school policy on parents working with their children in class. *The Journal of Educational Research*, 98(5), 281-298.
- Lehrer, R., & Schumow, L. (1997). Aligning the construction zones of parents and teachers for mathematics reform. *Cognition and Instruction*, 15(1), 41-83.
- Lin, P. (2002). On enhancing teachers' knowledge by constructing cases in classrooms. *Journal of Mathematics Teacher Education*, 5, 317-349.
- Lipka, J. (1994). Culturally negotiated schooling; Toward a Yup'ik mathematics. *Journal of American Indian Education*, 33 (Spring), 14-30.
- Little, J.W. (1999). Organizing schools for teacher learning, In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 233-262). San Francisco: Jossey Bass.
- Lubienski, S. T. (2002). Research, reform, and equity in US mathematics education. *Mathematical Thinking and Learning*, 4(2&3), 103-125.
- McBride, B. (1991). Parent education and support programs for fathers: Outcome effects on parent involvement. *Early Childhood Development and Care*, 67, 73-85.
- McClain, K., & Cobb, P. (2004). The critical role of institutional context in teacher development. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 281-288). Bergen: PME.
- McNamara, O., Hustler, D., Stronach, I., Rodrigo, M., Beresford, E., & Botcherby, S. (2000). Room to manoeuvre: Mobilising the 'active partner' in home-school relations. *British Educational Research Journal*, 26(4), 473-491.
- Meaney, T. (2001). An indigenous community doing mathematics curriculum development *Mathematics Education Research Journal*, 13(1), 3-14.
- Mehan, H., Hubbard, L., & Villanueva, I. (1994). Forming academic identities: Accommodation without assimilation among involuntary minorities. *Anthropology and Education Quarterly*, 25, 91-117.
- Meredith (2005). *Adding value? The complex question of mathematics homework*. Unpublished Research Project, Massey University, Palmerston North, New Zealand.
- Merttens, R. (1999). Family numeracy. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 78-90). Buckingham: Open University Press.
- Merttens, R., & Vass, J. (1993). *Partnerships in maths: Parents and schools*. London: Falmer.
- Millett, A., & Johnson, D. (2004). The role of the mathematics co-ordinator. In A. Millett, M. Brown, & M. Askew (Eds.), *Primary mathematics and the developing professional* (pp. 19-54). Dordrecht: Kluwer.
- Millett, A., Brown, M., & Askew, M (Eds.) (2004). *Primary mathematics and the developing professional*. Dordrecht: Kluwer.
- Ministry of Education. (2001). *Curriculum update 45: The numeracy story*. Wellington: Author.
- Ministry of Education. (2004). *Help your child to develop numeracy ... What you do counts! Numeracy Projects Years 3-6*. Wellington: Author.
- Ministry of Education. (2005). *Me mahi tahi kōrua ko tō tamaiti, kia whakawhanake i te mahi pāngarau ... Ahakoā te mahi, ka whai hua! Te Poutama Tau Years 1-6*. Wellington: Author.
- OECD. (2004). *Program for International Student Assessment. Learning for tomorrow's world: First results from PISA 2003* (OECD Publications No. 53799 2004). Paris: Organisation for Economic Co-operation and Development.
- Peressini, D. D. (1998). What's all the fuss about involving parents in mathematics education? *Teaching Children Mathematics*, 4(6), 320-325.
- Peters, S. (1994). *Mathematics intervention with seven-year-old*. Unpublished master's thesis, Waikato University, Hamilton, New Zealand.
- Peters, S. (1998). Gathering and sharing information on young children within the context of the transition to school. *SAMEpapers*, pp. 107-122.
- Pritchard, R. (2004). Investigating parental attitudes and beliefs in mathematics education. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010*. (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 478-485). Sydney: MERGA.
- Rosenholtz, S. J. (1989). *Teachers' workplace: The social organisation of schools*. New York: Longman.
- Salway, L., & Winter, J. (2003). Home school knowledge exchange project- Experiences from the first two years. *Proceedings of the British Society for Research into Learning Mathematics*, 2, 97-102.
- Sanchez, R.P., & Baquedano, M. M. (1993). *Curriculum of the home and mathematics achievement*. Paper presented at the Fifth Annual International Roundtable on Families, Communities, Schools, and Children's Learning, Atlanta.
- Savell, J. (1998). *Using parent newsletters to enhance junior primary school mathematics*. Unpublished master's thesis,

Massey University, Palmerston North, New Zealand.

- Savell, J., & Anthony, G. (2000). Crossing the home—school boundary. *New Zealand Research in Early Childhood Education*, 3, 51-65.
- Shaver, A. V., & Walls, R. T. (1998). Effect of parent involvement on student reading and mathematics achievement. *Journal of Research and Development in Education*, 31, 90-97.
- Sheldon, S. B. & Epstein, J. L. (2005). Involvement counts: Family and community partnerships and mathematics achievement. *The Journal of Educational Research*, 98(4), 196-206.
- Sherin, M.G., & Han, S.Y. (2004). Teaching learning in the context of a video club. *Teaching and Teacher Education*, 20, 163-183.
- Shulman, L., & Shulman, J. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies*, 36(2), 257-271.
- Smith, G. (1990). Ki te whai ao, ki te ao marama: Crises and change in Māori education. In A. Jones, G. McCulloch, J. Marshall, G. H. Smith, & L. Tuhiwai-Smith (Eds.), *Myths and realities: Schooling in New Zealand* (pp. 123-155). Palmerston North: Dunmore Press.
- Spillane, J. (1999). External reform initiatives and teachers' efforts to reconstruct their practice: The mediating role of teachers' zones of enactment. *Journal of Curriculum Studies*, 31(2), 143-175.
- Spillane, J. (2005). Primary school leadership practice: How the subject matters. *School Leadership and Management*, 25(4), 383-397.
- Starkey, P., & Klein, A. (2000). Fostering parental support for children's mathematical development: An intervention with Head Start families. *Early Education and Development*, 11, 659-680.
- Steinberg, R. M., Empson, S. B., & Carpenter, T. P. (2004). Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education*, 7, 237-267.
- Street, B., Baker, D., & Tomlin, A. (2005). *Navigating numeracies: Home/school numeracy practices*. Dordrecht: Springer.
- Sukon, K. S., & Jawahir, R. (2005). Influence of home-related factors in Numeracy performance of fourth-grade children in Mauritius. *International Journal of Educational Development*, 25, 547-556.
- Tiedemann, J. (2000). Parents' gender stereotypes and teachers' beliefs as predictors of children's concept of their mathematical ability in elementary school. *Journal of Educational Psychology*, 92(1), 144-151.
- Tizard, B., & Hughes, M. (1984). *Young children learning*. London: Fontana.
- Trinick, T. (2005). Te Poutama Tau: A case study of two schools. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 80-88). Wellington: Ministry of Education.
- Tudge, J., & Doucet, F. (2004). Early mathematical experiences: Observing young Black and White children's everyday activities. *Early Childhood Research Quarterly*, 19(1), 21-39.
- Walshaw, M. (1995). *The implementation of "Mathematics in the New Zealand Curriculum"*. Unpublished master's thesis, Massey University, Palmerston North, New Zealand.
- Walshaw, M. (2006). Girls' workplace destinations in a changed social landscape: Girls and their mothers talk. *British Journal of Sociology of Education*, (27) 5, 555-567.
- Walshaw, M., & Anthony, G. (2006). Numeracy Reform in New Zealand: Factors that Influence Classroom Enactment. *Proceedings of the 29th conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 361-368). Prague: PME.
- Wang, J., & Lin, E. (2005). Comparative studies on US and Chinese mathematics learning and the implications for Standards-based mathematics teaching reform. *Educational Researcher*, 34(5), 3-13.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.
- Wilson, S.M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. In A. Iran-Nejad & P.D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 173-210). Washington, DC: American Educational Research Association.
- Wood, J.J. (2003). *Professional development in mathematics for primary teachers*. Unpublished master's thesis, Massey University, Palmerston North, New Zealand.
- Young-Loveridge, J. M. (1989). The relationship between children's home experiences and their mathematical skills on entry to school. *Early Childhood Development and Care*, 43, 43-59.
- Young-Loveridge, J. M. (2004). Effects on early numeracy of a program using number books and games. *Early Childhood Quarterly*, 19, 82-98.

7. A Fraction of the Answer

Introduction

The evidence-based discussions in the preceding chapters serve to highlight the complexity of the learning and teaching process. In this chapter, we present extracts, which we call CASEs, from research studies situated in the teaching and learning of fractions. By grouping the CASEs around the domain of fractions (we include proportion and ratio within this domain), we hope to weave the threads of the preceding chapters into a whole that will be particularly helpful for initial and ongoing teacher education.

In presenting these CASEs, our aim is to capture the principal activities of teaching: (a) creating and supporting a learning community; (b) analysing and building on students' existing conceptions; (c) making sense of the mathematical ideas to be taught; and (d) selecting/using tasks that foster students' conceptual advances. At the same time, we also attempt to capture some of the principal activities of learning: (a) participating in a range of mathematical practices; (b) reflecting on one's learning processes; and (c) collaborating with others in sense making. This approach stems from our view that successful teaching involves processes that teachers and students have to work through together. "Teaching is not something that is merely 'done' to learners. [Students] are active participants in classrooms and the transactional nature of teaching and learning means that [students] as much as teachers shape the development of lessons" (Askew, in press).

Why have we chosen fractions? As many teachers know from first-hand experience, fractions are one of the most complex mathematical domains that students encounter during their school years (Davis, Hunting, & Pearn, 1993). The difficulties that New Zealand students experience with fractions have been highlighted in reports evaluating the Numeracy Development Project and Te Poutama Tau (e.g., Christensen, 2004; [Young-Loveridge, 2005](#)).

Fractions constitute a body of knowledge considered essential not only for higher mathematics but also for everyday life. It is our understanding of fractions that makes it possible for us to contemplate 'how much?' as opposed to 'how many?' Understanding of fractions signals a shift from reasoning additively to reasoning multiplicatively. This development begins when young children experience sharing situations (Hunting & Sharpley, 1988) and—in the school context—culminates with ratio and proportional thinking (Thompson & Saldanha, 2003).

While early research into students' understanding of fractions tended to document their poor performance and their misconceptions (e.g., Hart, 1988), more recent research has explored the possibilities that exist within instructional settings for the creation and development of fractional knowledge. We have selected CASEs from research to exemplify aspects of effective pedagogy: pedagogy that makes a difference to student outcomes. In some studies, the differences are measured by pre- and post-tests, sometimes by contrast with the performance of students in other classes (e.g., CASE 4). In other CASEs, change is indicated by the success of one or two learners who are beginning to participate in mathematical practices, resolve cognitive conflict, grow in understanding, or develop metacognitive awareness (e.g., CASE 7).

In each of the selected CASEs, opportunity and space for learning are key factors. A supportive environment and established social and sociomathematical norms encourage students to reflect on their learning with an expectation that they will be able to make sense of the mathematics involved. Teacher awareness and responsiveness, developed through critical listening to student responses and thinking, are also key features of the selected CASEs.

In selecting exemplary CASEs, it is not our intention to minimise or hide the everyday complexity of the teaching and learning environment. Several of the CASEs are from studies that involve teacher professional development. The full reports of these studies often include teacher commentary on the real-time challenges associated with task selection, managing group work,

and facilitating meaningful discussion—including dealing with the unintelligible in classroom discussions. O'Connor (2001) acknowledges that teachers, when presented with accounts of classroom discussions conducted by talented and skilful teachers, are often concerned with what is *not* in the transcript: “What were the other students doing while the teacher attended to a particular group of students?”; “Why didn’t she just tell them about x or y ?”; or “Why did she ignore that student’s incorrect answer?” So, in reviewing each of the following CASEs, the reader might reasonably ask, “What is left out of the account?”

In each CASE, the presented episodes are adapted from research accounts and are therefore necessarily selective and somewhat brief. Although chosen to exemplify quality pedagogical practices, the CASEs are not intended, by themselves, to provide a list of ‘good ideas’ for teaching fractions. Rather, the expectation is that they will stimulate critical thought about effective pedagogical practices for mathematics in general and for fractions in particular.

Building on learners’ prior knowledge and experiences

Prior to formal schooling, many opportunities for learning mathematics can be found in children’s everyday experiences—in the home, the community, or centre. CASE 1 exemplifies the supporting role of the adult. Building on a relationship of trust, the adult is able to engage with the child in sustained, shared thinking involving an imaginative and experientially real context.

CASE 1: Cookies

(from Sharp, Garofalo, and Adams, 2002)

Mathematics teaching for diverse learners:

- demands an ethic of care;
- creates a space for the individual and the collective;
- provides opportunities for children to explore mathematics through a range of imaginative and real-world contexts;
- provides for both planned and spontaneous/informal learning.

Through experiences such as ‘sharing’, young children develop intuitive fractional knowledge in which they combine thought, informal language, and images (Kieren, 1988).

Targeted outcomes

Informal knowledge of fractions developed through context-based sharing situations.

Learning context

Leah, who is almost five, and Joe, an adult, often play games while driving to Leah’s preschool in their truck. Leah is engaged in a sustained shared-thinking episode as part of a game that she regularly plays with Joe, which she calls ‘Kids and Cookies’.

Task and activity

- Joe: Hey Leah, what do you want to play today?
- Leah: Let’s play Kids and Cookies.
- Joe: OK. What if you had 4 cookies and 3 kids? How would you share them?
- Leah: One, one, one, and then there is one left. Then they each get one third, one third, one third.
- Joe: So, how much does each kid get?
- Leah: They get one whole one and one third.
- Joe: What if you had 5 cookies and 3 kids? How could you share the cookies?
- Leah: One, one, one. Then there’s two more left. OK. Then they get a third, a third, a third, and then a third, a third, a third.

Joe: So how much do they each get?

Leah: They get one whole and two thirds.

Joe: What if you had 7 cookies and 4 kids?

Leah: That's a hard one, maybe I can't do it.

Joe: Think about what you did to solve the other two.

Leah: Whole, whole, whole, whole, then there's three more left. Um, three more cookies left. Then you break up one into halves, then there are two left. And, another into half, half. Break the last one into quarter, quarter, quarter, quarter.

Joe: Great! How much does each kid get?

Leah: One whole, one half, and one quarter.

Joe and Leah played the game for a few minutes, several times a week, with Joe varying the numbers so that Leah was able to resolve more and more complex situations. Sometimes Joe would ask Leah to find two ways to share the cookies (e.g., four cookies could be shared among six children by splitting each cookie into sixths and giving each child four-sixths, or by giving each one-half of a cookie and one-sixth).

Learner outcomes

By comparing different solution strategies for problems, Leah developed an understanding of equivalent fractions. With repeated exposure to the game, Leah's ideas were both valued and challenged. She was able to build on her existing conceptual understanding of fractions and operational sense of whole number to develop a procedure that would foreground operations with fractions.

Quality pedagogy

Leah's growth in understanding of fractions was assisted through:

- interaction with a supportive adult. Joe's questions incorporated ideas about sharing and used Leah's informal language.
- activities situated in experientially relevant contexts. Leah's conceptual knowledge of fractions grew from encounters with whole-number division.
- instruction that built on her informal knowledge. This gave Leah access to participation, allowing her to make contributions that were personally meaningful.

Bridge to school

Children may well enter school with a rich bank of informal or intuitive understanding of rational number concepts and procedures, based on their activities in their personal environment. It is through these activities that students develop thinking tools and imagery for the construction of important knowledge about rational numbers.

Positioning participants as proficient learners

In chapter 4, we found that teachers who produce effective classroom communities seek to develop interrelationships that create spaces for students to develop their mathematical identities. In caring, teachers developed a culture that did not minimise individuals' experiences and contributions within the classroom. Students were trusted with responsibility for themselves and their learning and were provided with opportunities to exercise this responsibility (Angier & Povey, 1999). In CASE 2, the focus is on two low-performing students' experiences in a series of lessons based on equal-sharing fraction tasks. Empson (1999) takes the data from a successful classroom intervention study involving early fraction learning and reanalyses it from a participation perspective, in order to unpack how it is that these two students profited from their classroom experience, "not despite the cognitive or social skill they may have lacked but because of the way their teacher orchestrated their participation in solving and discussing problems" (p. 305).

CASE 2: Low-performing students' participation

(from Empson, 2003)

Mathematics teaching for diverse learners:

- demands an ethic of care;
- demands teacher content knowledge and pedagogical content knowledge and reflecting-in-action;
- involves explicit instructional discourse;
- provides opportunities to explore mathematics through a range of relevant contexts;
- provides tasks that are problematic and have a mathematical focus;
- provides opportunities to resolve cognitive conflicts and problematic reasoning.

This case presents two low-performing students' experiences in a first grade classroom. Explanations for student gains in fraction knowledge are analysed from the perspective of the dynamics of the instructional interactions and their consequences for the students.

Targeted learning outcome

Gains in Patrick and Pho's understanding of fractions.

Learning context

The data are drawn from a larger case study (Empson, 1999) of a US grade 1 class involved in a five-week unit on fractions. The instruction was organised around eliciting and building on children's informal knowledge of equal-sharing situations.

The students

Two students, Patrick and Pho, were assessed in a clinical interview as knowing least about fractions, when compared with their classmates, both at the beginning and at the end of the study. Patrick, from a middle-class family, had been informally identified by school personnel as having trouble focusing on academic tasks. Because he was being withdrawn to participate in an alternative programme, he was present for a total of 9 of the 15 fractions lessons. Pho's family spoke English as a second language. He was present for all 15 lessons.

Pedagogical approach

Assisting students to resolve problematic reasoning is an issue faced by all teachers, especially so in fractions (Thomson & Salanha, 2004). Empson provides several vignettes that illustrate how the teacher, Ms. K, played a key role in orchestrating the participation of Patrick and Pho in the mathematical practices of the classroom community. The participant frameworks that emerged in her classroom supported Patrick and Pho to participate, when possible, in resolving problematic aspects of their reasoning.

This vignette involves a group situation. Students are solving the problem "There are 2 horses in the field. If they have one bushel with 9 apples in, how many apples would each horse get?" The interaction that follows begins after the first student explained his strategy to the group, including how he split the extra apple in half. When Ms. K called on Pho to report his solution, a predicament emerged with Pho's declaration that he could not give the horses the extra apple:

- 1 Pho: There's one more left, but I can't give them this.
- 2 Ms. K: So what are you going to do with that one left?
- 3 Pho: Uh.
- 4 Ms. K: Anybody got an idea? What can we do with that other one?
- 5 James: I know, I know.
- 6 Ms. K: (To Pho) Do you know what you could do with this one?
- 7 Pho: (Shakes head no)
- 8–9 Ms. K: No. You're not sure. Could anyone share with him what he could maybe do with that other one? James?
- 10 James: Cut it in half.
- 11 Ms. K: Could you cut it in half, Pho?

- 12 Pho: (Inaudible)
- 13 Ms. K: And then how much would each horse get?
- 14 Pho: It still wouldn't work.
- 15 Ms. K: Why won't it work?
- 16 Pho: 'Cause, there still is one more left.
- 17 Ms. K: OK. How much is this horse [pointing to his cubes] going to get?
- 18 Pho: Four.
- 19 Ms. K: How much is this horse [pointing to second horse] going to get?
- 20 Pho: Four.
- 21 Ms. K: James, he says it is still not going to work. Would you please go down—
- 22 Pho: (Interrupting) Because four and four is eight.
- 23–24 James: [To Pho.] You have eight so far. Just cut that (extra apple) in half, and then each one would get eight (sic) and a half.
- 25 Pho: I still can't get it.

In this episode Pho was positioned as a problem solver. Throughout the episode the teacher elicited help from the other children. The distribution of the role of problem solver enabled the other children to act as potential sources for problem-solving ideas. Pho, however, retained the authority to accept or reject the ideas offered based on his understanding of that idea, and in this case he was seen to reject a mathematically legitimate idea proposed by James. While the rest of the group solved a new problem, Ms. K worked with Patrick and Pho to help them arrive at the solution of partitioning the extra apple in half and verbalising how that quantity related to the whole apples (see the following interaction with Patrick).

- 1 Patrick: Hmm.
- 2 Ms. K: What did the other kids say we could do with this other apple?
- 3 Patrick: Split it in half. But we can't do it.
- 4–5 Ms. K: Well, let's pretend it's [holds up a linking cube] an apple. If it's an apple could we cut it in half? [Patrick agrees.] ... So how many apples is each horse gonna get in this now?
- 6 Patrick: One half.
- 7–8 Ms. K: OK. And they're gonna get these [indicating four linking cubes each] ... How much would they get?
- 9 Patrick: Five.
- 10 Ms. K: How did you figure out five apples?
- 11–12 Patrick: Because if we cut this [extra linking cube] in half they would each get five apples.
- 13–15 Ms. K: Show me. [Patrick counts four single apples each, and on the extra cube, two apples (one for each half).] Is this last piece they're gonna get a whole apple or is that gonna be a half apple?
- 16 Patrick: Half apple.
- 17 Ms. K: Are these [four linking cubes] whole apples or are they half apples?
- 18 Patrick: Whole.
- 19 Ms. K: So how many whole, big apples are they gonna get?
- 20 Patrick: Four.
- 21 Ms. K: OK. And how many half apples?
- 22 Patrick: One. [Patrick writes '4 1' on his paper.]

The interactions were structured so that Patrick and Pho revoiced the key ideas introduced by other children in the earlier exchange. This enabled the two boys to be responsible for evaluating potentially useful mathematical ideas and to begin to make the ideas their own. Throughout the interactions the teacher's use of "what?" and "how?" questions directed the boys' attention to the mathematically critical aspects of the solution—that is, the difference between whole apples and fractional apples as amounts. Physical materials were used as a support for thinking, rather than a literal representation (see Higgins, 2005).

In this and other episodes provided by Empson, we can see that Ms. K oriented Patrick and Pho's participation in instruction towards problem-solving practices and towards taking on an authoritative role. A significant factor in gaining their participation was the acknowledgement and building on the task-based contribution that each boy was able to make. Empson noted that Patrick and Pho had ideas about how to solve almost all of the problems. While these ideas were sometimes partially formed, ambiguously stated, or notationally unsophisticated, they were, with the assistance from Ms. K's scaffolding, able to be treated as part of the pool of ideas for solving the problem. The contributions of Patrick and Pho were accepted or rejected based on mathematical reasons supplied by the learning community.

In the following episode, Pho is positioned as a mathematical authority. This is the first time children are asked to solve an equal-sharing problem involving partition into thirds, a partition that, from a geometric perspective, is harder to make than partitions involving repeated halving (Pothier & Sawada, 1983): "Three children want to share seven candy bars so that everyone gets the same amount. How much would each child get?"

- 1–2 Marie: Because there's—you can't leave one over [i.e., if you make fourths, you will have an extra piece], so if you cut this one [extra cube] in half—
- 3–5 Ms. K: But Marie, look what Pho did [Marie looks at Pho's paper]. Pho, she says you can't split that in three. You think that's right? [Inaudible answer from Pho.] What do you think guys?
- 6–7 Tim: If you split in three, then you would get a half that remains, a half of a candy bar that's still there [Empson noted: he may mean a fourth].
- 8 Ms. K: If you split it in threes?
- 9 Marie: That's what I'm talking about. You still get a quarter left.
- 10 Ms. K: Pho, they said that you can't split it in threes.
- 11 Tim: You can split it in threes but you have a half left.
- 12–13 Ms. K: You'll have a half? Look at how he split it, Tim. Does he have a half left over? [Lev goes over to look at Pho's paper; Tim looks too.]
- 14 Tim: No.
- 15 Ms. K: [To Pho] Do you have a half left over?
- 16 Kaitlin: No.
- 17 Pho: No.
- 18 Tim: That's because of the way you cut it.
- 19 Ms. K: He cut it differently didn't he? Would that work, Tim, or not?
- 20 Tim: It would work.
- 21 Pho: I'll draw it over.
- 22 Ms. K: Would you draw it bigger so people can see.

In this episode Pho was animated by the teacher making a mathematical claim in opposition to another apparently reasonable mathematical claim. In managing the distributed argumentation the teacher created opportunities for Pho to respond directly to Tim and Marie's claim about the impossibility of splitting the bar in three. By relaying their statements to Pho, the teacher effectively scaffolded Pho in this role. In her contribution in lines 12–13, the teacher directed Tim to the part of Pho's diagram illustrating the main claim, thus modelling a move Pho could have made himself. Empson claims that the explicit positioning of the competing ideas assisted the students to resolve the conflicting representations and elevated Pho's solution to the status of a defensible claim of value (lines 20–21). Ultimately, thirds became an acceptable partition in the class. In a later interaction, Patrick was also positioned in the role of author of a partitioning strategy that formed a critical piece in an argument about equivalent fractions.

Learner outcomes

Comparison of the pre- and post-interview results documents Patrick's and Pho's gains in understanding. Throughout the series of lessons Ms. K positioned Patrick and Pho to make contributions to group discussions that enabled them to be animated in identity-enhancing ways—as problem solvers, solution reporters, and claim defenders. These students, although low attainers, were able to successfully engage in problem-solving processes, communicate their thinking, and build complex arguments about mathematical relationships—practices that are essential for learning and doing mathematics (Watson & De Geest, 2005).

Quality pedagogy

Empson proffers three main factors to explain how these two boys, who clearly struggled in mathematics, were supported to participate profitably in Ms. K's classroom:

- Both Patrick and Pho were treated as competent in ways that were valued in the classroom community of practice. Contributions, based on their informal ideas, enabled the boys to be positioned as legitimate knowers (Lave & Wenger, 1991).
- The mathematical tasks posed to Patrick and Pho allowed them to make use of prior knowledge to generate new strategies. These semantically rich problems afforded a variety of strategies—in Pho and Patrick's case, partial strategies—which provided a basis for productive interactions between teacher and students.
- Patrick and Pho had multiple opportunities to learn the value of their ideas and the practices entailed in developing and articulating those ideas. They were at all times regarded as mathematical thinkers and doers.

“Although Ms. K knew that Patrick and Pho did not, generally, understand as much about fractions as the other students, they were not animated as children who did not understand. They were instead animated as children who engage in the practice of mathematics, and consequently, as children who understood mathematics. This positioning positively affected their participation and learning” (p. 339).

Challenging tasks with a mathematical focus

As we have seen in chapter 5, high-involvement teachers like Ms. K in CASE 2 typically present challenge as desirable, treat errors as informational, provide feedback on progress, and help students resolve conflicting reasoning. High-involvement classrooms were the focus of Turner and colleagues' (1998) study. A sixth grade teacher in this study, Ms. Adams, took a lesson on fractions and the researchers noted how she regulated the challenge of the task to match her students. Instead of altering the task goal, she adjusted the instructions until her students' skills matched the challenge. For example, to avoid reducing the overall complexity or compromising the integrity of the task, Ms. Adams modelled strategies such as reducing fractions expressed with large numbers to equivalent fractions expressed with smaller numbers. In contrast, when a low-involvement teacher in the same study wanted her students to convert $\frac{161}{184}$ to a percentage, she took over the task: “Most of us don't remember this, but if we want to turn this into a decimal, we would divide 184 into 161.”

Students in low-involvement classrooms typically report feelings of boredom and less positive affect, and that their skills exceed the challenges provided. Houssart's (2002) UK study of a class of nine- and ten-year-old 'low attainers' illustrates how task challenge and related mathematical focus can influence students' performance. During a six-week unit on fractions, the children in the research class completed 18 worksheets requiring low-level identification and colouring of fractions of shapes. Houssart's observations of three boys revealed that each attempted, both privately and publicly, to extend ideas introduced by the teacher. For example, in a folding exercise to demonstrate halves, the boys speculated on whether thirds could be made in a similar way. The teacher, however, failed to acknowledge the students' attempts to extend the task. As the unit progressed, the boys commented on the ease of the work and their behaviour became more disruptive and off-task. The teacher's reaction was to express disappointment with the class and focus on structured worksheets. Henceforth, the instructional focus became task completion rather than the development or demonstration of mathematical understanding. As a consequence of the written work being too easy, the boys worked quickly without listening to the teacher, re-reading instructions, or seeking help. The fact that the work was so easy became, according to Houssart, a progressively stronger factor in the boys' apparent failure. By the end of the unit, these boys were failing to complete much of the easy work and they were communicating their dissent publicly.

Superficially, this could be seen merely as bad behaviour and failure to cooperate. However, the crucial point is that the behaviour was a result of shifting classroom

norms and practices. Non-challenging mathematics produced challenging behaviours. In terms of performance on formal written tasks it also produced failure, though oral response provided contrary indications. (Houssart, 2002, p. 210)

The study raises two significant issues for pedagogical practice. First, when gathering evidence about student learning, teachers need to consider evidence from a range of tasks, both written and oral. And second, more significantly, with regard to task challenge, Houssart's work contests the view that task simplification and repetition is appropriate for low attainers.

In contrast, tasks that present higher-level demands use procedures but in a way that build connections to the mathematical meaning (Stein et al., 1996). CASE 3 presents an example of a challenging task that arose spontaneously during a class activity. The mathematically rich activity invited student exploration and challenge at a range of levels appropriate to all students in the class.

CASE 3: Flags

(from Kieren, Davis, and Mason, 1996)

Mathematics teaching for diverse learners:

- involves explicit instructional discourse;
- creates a space for the individual and the collective;
- demands teacher content and pedagogical content knowledge and reflecting-in-action;
- provides opportunities for cognitive engagement and a press for understanding;
- utilises tools as learning supports.

This case describes students' exploration of fraction concepts using a student-generated activity, 'fraction flags'. The case is framed from the perspective that students benefit from engaging in mathematically rich activity—activity that invites exploration and conjecture. In addition, the teacher selects classroom learning activities that are responsive to students' knowledge and interests.

Targeted learning outcomes

Learners view fractions as representing additive quantities and as showing multiplicative relationships.

Learning context

This activity is derived from an exploration by two 12-year-old students, Tanya and Ellen, of the pieces from a 'pizza fractions' kit. The six-week unit that gave rise to the 'flags' activity was developed around various paper-manipulating activities: folding, cutting, comparing, rearranging, and assembling. The tools used to support learning included physical manipulation of units and fractional sub-units built from paper, mental actions on the images of fractions constructed by students, and the verbal and symbolic expressions of actions, observations, and justification.

Task and student activity

The fractions unit centred on providing opportunities for students to build their own ideas of fractions. A key representation for investigating multiplicative notions involved folding units (standard pieces of paper) and sub-units into various numbers of parts. For example, by exploration, repeated folding could generate thirty-twoths. Another representation, the 'pizza fractions' kit, (assorted rectangular fractional pieces including wholes, halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths, as in figure 7.1) offers students the opportunity to develop an additive, quantitative sense of rational numbers.

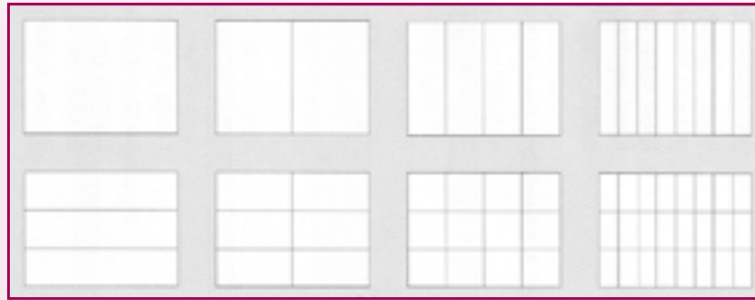


Fig. 7.1. The 'pizza fractions' kit

Activities based around the pizza kit included student-generated pizza orders, such as $\frac{3}{4} + \frac{2}{3} + \frac{5}{12}$. Students' arrangement of the pieces supported the development of images and understandings. Several strategies were used by students to present their results, including drawing pictures, writing out fraction phrases and sentences, and reproducing summary charts.

Within this context, Kurt, playing with some pieces from the kit while waiting for the teacher, created a flag (fig. 7.2). Prompted by another student's enquiry as to how much of the paper was left uncovered, the teacher structured a new setting for further fraction problems.

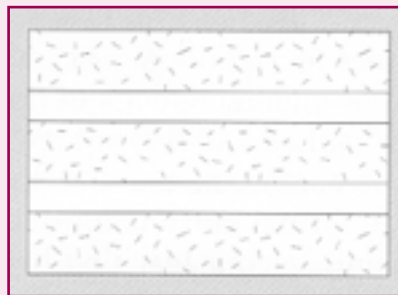


Fig. 7.2. Kurt's flag

The 'fraction flags' activity was introduced to the class in this way:

Take a half piece, a twelfth piece, and two eighth pieces from your kit and make this flag (see fig. 7.3). Show it to your partner and make up some fraction questions about the flag. For example, is more of the base covered or uncovered? How much more? Use pieces from your kit to make up your own flag. Once you have done this, make up some fraction questions about it. Try to have your partners or other students in class answer your questions. All members of your group should be ready to discuss your flags and questions with the rest of the class. Remember, try to make interesting flags, but also make flags so that you can ask good fraction questions about them.

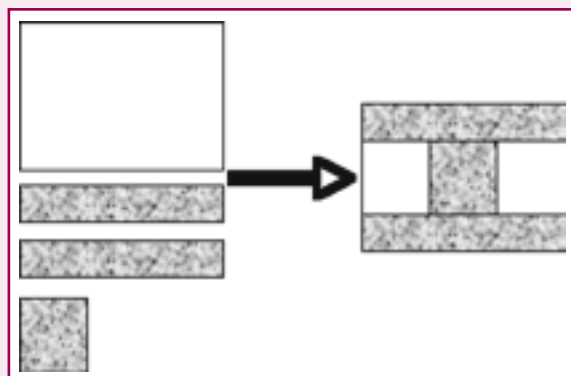


Fig. 7.3. 'Fraction flag' questions

Student outcomes

When left to design their own questions within contexts that were of interest, students developed situations that were personally challenging. For example, Tanya and Ellen made a fraction flag (fig. 7.4) by taking a half-sheet of paper and arranging smaller pieces on it, then worked out how much of the whole sheet of paper was covered.

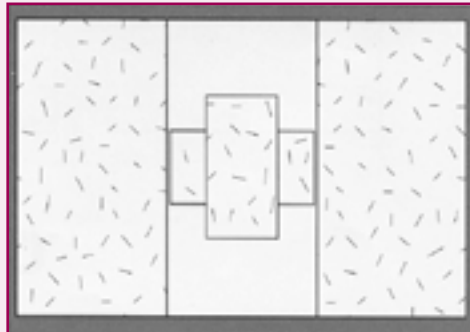


Fig. 7.4. Tanya and Ellen's fraction flag

- Ellen: The edge parts are easy—that's just two-sixths [of a whole sheet]—but the middle part is hard.
- Tanya: That's because it's a twenty-fourth on top of a twenty-fourth.
- Ellen: I can see the twenty-fourth in the middle, but I don't get the two little pieces on its sides.
- Tanya: [Sliding over the top twenty-fourth piece] Oh, I get it. Those two side parts make a half of a twenty-fourth together, and that's a forty-eighth.
- Ellen: Okay! So the total covered on the flag is two-sixths plus one twenty-fourth plus half a twenty-fourth.
- Tanya: Right! So that's four, eight, nine-and-a half twenty-fourths. What's that in forty-eighths?

The two girls solved the coverage problem by rearranging the pieces so that they could "see" the amounts involved. Although the calculation was informal, it amounted to $\frac{2}{6} + \frac{1}{24} + \frac{1}{2}(\frac{1}{24}) = \frac{9.5}{24} = \frac{19}{48}$ —a significantly challenging task for students considered to be average achievers in mathematics.

The flag activity gave students an opportunity both to develop calculation skills and to invent situations that required more flexible strategies.

Students were positioned as mathematical doers and thinkers: "It's good. It's a little challenging; It's not boring to do" [Ellen]; "A way of asking questions about the world" [Greg].

Quality pedagogy

Enactment of the flag activity highlights factors that supported productive learning:

- Task design was premised on a combination of a predetermined learning trajectory based on teacher knowledge of fractions and significant milestones.
- The teacher responded flexibly to students' interest and learning needs. The introduction of the flag activity was a direct result of the teacher attending to the learners—learning from them—to structure responsively a more effective learning environment.
- The flag activity is a mathematically rich activity that invited exploration and conjecture while offering opportunities for personal and aesthetic expression.
- The flag setting allowed students to interact with one another, with the teacher, and with the objects in their world. The resulting conversations enabled the teacher to observe and respect the diversity in students' fractional thinking and in their expression.
- Class discussion supported the introduction and encouragement of the use of standard fraction language and symbols.
- Students were able to create problems for themselves that were appropriate to their own levels of understanding.

These pedagogic strategies involve "not simply helping students to learn but, more fundamentally, learning from the learners" (p. 19).

Using tools to support learners' mathematical thinking

As we have seen in the previous CASEs, children's exploration of fractions can usefully involve a range of representational tools including drawings and diagrams, symbols, and manipulatives such as paper rectangles. These representations assist learners to focus on certain key features of fractions.

As discussed in chapter 5, tools can also support students' strategic thinking and help make their solution strategies visible to others. In CASE 2, **Empson (2003)** noted that allowing children to choose their own tools and make their own representations to solve equal-sharing problems fosters "an interesting diversity of thinking, which can contribute to richer understanding of the mathematics of fractions" (p. 35). For example, when using part-whole representations, sixths can be drawn in three ways: by making halves and partitioning each into thirds, by making thirds and partitioning each into halves; and by making a guess about how big a sixth is and partitioning the pieces one by one. Each method supports different mathematical representations of equivalence. The first justifies the equivalence of $\frac{1}{2}$ and $\frac{3}{6}$; the second justifies the equivalence of $\frac{1}{3}$ and $\frac{2}{6}$; and the third lends itself to the idea that $\frac{6}{6}$ is equivalent to 1.

CASE 4 examines the use of a real context, sharing cakes, in which students move from having a model 'for' division to a model 'of' division. Division of fractions is readily acknowledged as the most complex of the arithmetical operations (Ma, 1999). Fraction division can be explained as an extension of whole-number division categorisations—measurement division, partitive division, and the inverse of a Cartesian product. Division as the determination of a unit rate and division as the inverse of multiplication are two further important fraction-division interpretations (Sinicrope, Mick & Kolb, 2002). Traditionally, children have been taught division as a rule-based procedure, with little attempt to ground this procedure in a meaningful context. The researchers in this and several other studies claim that building connections with informal knowledge and encouraging students to explore multiple explanations allows students to develop a robust understanding of rational numbers.

CASE 4: Dividing the Cake

(from Sarage, 1992)

Mathematics teaching for diverse learners:

- involves respectful exchange of ideas;
- provides opportunities for children to resolve cognitive conflict;
- provides for both planned and spontaneous/informal learning;
- utilises tools as learning supports;
- provides opportunities for students to problematise activities based in realistic contexts;
- builds on students' prior knowledge and experiences.

Targeted learning outcomes

Solving problems involving division with fractions.

Learning context

A class discussion involving the solution of the problem "Four block cakes are to be divided into portions of three-fifths of a cake. How many portions are there?"

Student activity

Following exploration of the problem, the students were required to share their solution strategies with the class.

Vivian: I'll show my diagram [going to the board to demonstrate her solution]. See, here are the four cakes (fig. 7.5). And you can see that you get six portions out of them.

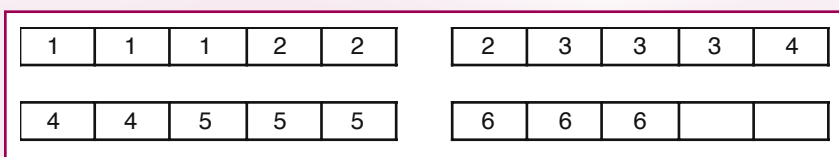


Fig. 7.5. Four cakes divided into portions of $\frac{3}{5}$.

Claire: What about the left overs?

Vivian: [After a glance at her diagram] Okay, so it's 6 and $\frac{2}{5}$.

Jonathan: Wait a minute. When I do the calculation I get 6 and $\frac{2}{3}$.

Vivian: But look at the picture. You can see that it's $\frac{2}{5}$ left over.

Seeing the significance of the contradiction, the teacher interrupts their debate to make sure the rest of the class sees it too. Instructed to work with a partner, the students puzzle over what to do with the two pieces of cake that are left over after six people take their portions. When the class is called together again, Carole attempts her group's explanation:

Carole: The problem asks about portions. You can say that there are 6 portions with $\frac{2}{5}$ of a cake left over or you can say that there are 6 and $\frac{2}{3}$ portions.

Sandy: Look at something else in Vivian's diagram. She started with 4 cakes. Then she cut each cake into 5 pieces. So she had $4 \times 5 = 20$ pieces. Then she grouped those pieces by threes since 3 pieces make up a portion. So she got $20 \div 3 = 6\frac{2}{3}$ portions. She multiplied by the denominator and divided by the numerator. Like in flip and multiply!

Eleanor: I see something else in that diagram. You've got $\frac{3}{5}$ of a cake equal to a portion. But you can see that each cake is one portion plus another $\frac{2}{3}$ of a portion. That is, each cake is $\frac{5}{3}$ of a portion. So when you want to find out how many portions there are in 4 cakes, you can divide by the size of each portion ($4 \div \frac{3}{5}$) or you can multiply by the number of portions per cake ($4 \times \frac{5}{3}$). Amazing!

Both Sandy and Eleanor are describing the meaning they now find in the usually mysterious rule for dividing by fractions.

Learner outcomes

The classroom discourse of enquiry encouraged students' struggle to resolve conflicts or confusions in their thinking. The discussion resulted in students' productive reorganisation of the mathematical ideas into more complex levels of understanding. By allowing these students to proceed with an explanation, even when their initial answer was wrong, the teacher fostered an expectation that the mathematical authority resides within the mathematical justification, to be shared and endorsed by both teacher and students. Sharing the locus of authority meant that students in this class were free to develop confidence in their own methods and their own monitoring skills when deciding whether something made sense. Rather than trying to uncover what the teacher wanted, students were "free to focus their attention on developing justification for their methods and solutions based on the logic of mathematics" (Hiebert et al., 1997, p. 41).

Quality pedagogy

In this episode, the teacher viewed the students' activity as meaningful—their errors were treated as building blocks to understanding. Conceptual understanding was supported by pedagogical practices that provided:

- opportunities for students to experience constructive doubt and conflict;
- opportunities for students to use familiar representations to develop, explain, and monitor their thinking;
- challenging contextual tasks that have a clear mathematics focus and purpose;
- a variety of tools to facilitate informal communication;
- opportunities for students to take the initiative and experience ownership in their learning;
- opportunities for students to engage in meaningful mathematical practices within a supportive learning community.

Task engagement and sense making

As we have seen in the previous cases, students' sense making is a key focus of their activities. Providing students with rich tasks within an enquiry-type environment may or may not produce the desired learning activity and outcome (Henningsen & Stein, 1997). In CASE 5, we see how the first assigned task, involving rate calculations, failed to provoke appropriate solution strategies. Exposing teacher indecision as to what to do in such a situation, the case documents a successful way forward.

CASE 5: Calculating Rates

(from Smith, 1998)

Mathematics teaching for diverse learners:

- involves respectful exchange of ideas;
- provides opportunities for children to resolve cognitive conflict;
- involves sequencing of tasks and provision of appropriate challenge;
- utilises appropriate tasks with a mathematical focus—e.g., extreme examples;
- provides opportunities for students to problematise activities based in realistic contexts;
- involves explicit instructional discourse.

In this case, the teacher's first attempt to stimulate students' solution strategies with a rich task only served to affirm existing misconceptions. The teacher needed to reconsider how to challenge these existing misconceptions with a new task.

Targeted learning outcomes

Use of rational equations to solve problems involving rates.

Learning context

Working in a senior secondary remedial mathematics course, the teacher introduced the unit on "rational equations" (p. 750) with the following 'Two Hands Are Better than One' problem:

Darlene and John Edinger were looking for someone to paint their front porch. They received several estimates. The two best estimates came from Michael and Tim. Michael said that he could do the job in 8 hours. Tim told the Edingers that he could complete the job in only 6 hours. Darlene and John wanted the porch painted as quickly as possible, so they decide to hire both men. Approximately how long should it take for Michael and Tim—working together—to complete the job?

Students were required to solve this problem in groups, justifying their solutions to each other. The teacher walked around the room, monitoring their progress.

Student activity

Although the reasoning processes differed in appearance, the group solutions were all based on the misconception that the two painters would work at a rate determined by $\frac{\text{average time}}{2}$. For example:

Group A

Michael = 8 hrs (2 people = $\frac{1}{2} \times 8 = 4$); Tim = 6 hrs (2 people = $\frac{1}{2} \times 6 = 3$). So, both together = $(3 + 4) \div 2 = 7 \div 2 = 3.5$ hrs.

Group D

$8 + 6 = 14$, 2 people = 7. The Edingers need the job done as quickly as possible, so they hired both boys. So you take the average of both and divide that number by 2 people. You get the time it will take 2 people to do the job, 3.5 hours.

Teacher reflections

The teacher listened in on a group discussion in which Sally put forward the following argument: "Well I know that the average is seven, and somehow you have to do this, since you are having both of them ... I know that seven is not the answer. But we have to find their time together, and I think we need to know the average to get it." The teacher reflected to himself:

I hadn't anticipated this approach! I don't really see how the average could be useful. Should I say something now? Should I let them continue to pursue this conjecture? I'm really not prepared to address this misconception at the moment. This is not what I was expecting. Everyone seems to agree with Sally. Isn't anyone going to question her conjecture? ... It looks like the whole group is buying into this idea ... Should I be the one to question it? ... What a mess!

A new problem

[After some consideration] the teacher constructed a new problem for which the 'average time $\div 2$ ' method was clearly not a sensible approach:

Suppose that Michael could complete the job in 10 hours and that Tim could complete the job in 2 hours. How long would it take the two men working together to complete the job?

Student activity

Groups initially applied the $\frac{\text{average time}}{2}$ algorithm to the new problem. But in some groups, the student reflections on the answer caused some questioning of this approach:

Kerri: I got three.

Hannah: I did too.

Lisa: Wait, you guys, this answer can't be right. It only takes Tim two hours to do it alone.

Tabitha: [attempting to explain this seemingly impossible answer] Maybe Michael slows Tim down—that could happen. When I work with someone who is real slow, it happens to me.

Lisa: No. Three is right. We did something wrong.

Kerri: [agreeing with Tabitha] Maybe Tim did slow down. Why don't the Edingers just hire Tim? Michael is too slow.

Teacher support

When the teacher returned to the whole-class discussion the students were asking good questions and were ready to move forward. They collectively confirmed their suspicion that the $\frac{\text{average time}}{2}$ algorithm was not appropriate. Realising that the students needed further guidance if they were to proceed, the teacher offered the following suggestion: "Consider each worker's rate per hour". The groups returned to the problem and successfully found the exact solution, $t = 1\frac{2}{3}$ hrs, for $\frac{1}{2} + \frac{1}{10} = \frac{1}{1}$ hrs [or $\frac{1}{2} + \frac{1}{10} = 1$].

Quality pedagogy

Factors that facilitated students' sense making with these rate problems included:

- encouragement for students to explain their solutions and develop their own sense of accuracy;
- use of teacher questions to elicit explanations and guide students toward persuasive justification of their solutions;
- a focus on conceptual rather than procedural content. The teacher-provided information focused students on the measurable attribute of the objects in the problem (rate per hour) and the relationships among quantities;
- use of extreme examples;
- respect of student ideas and a recognition that errors can be a useful starting point for effective discussion.

Respectful exchange of ideas

Research evidence presented in chapter 5 indicated that students' sense making involves a process of shared negotiation of meaning. In discussion-intensive learning environments, a high proportion of content development occurs through collective argumentation and group discussion. Sharing strategies and being involved in collaborative activities does not, however, necessarily result in students engaging in practices that support the development of

mathematical understanding. Shaping productive mathematical dialogue involves establishing classroom sociomathematical norms that are specific to students' mathematical activities (Cobb et al., 1993; Yackel & Cobb, 1996). In CASE 6, classroom exchanges centred on fraction tasks illustrate how sociomathematical norms govern classroom discussions and potential opportunities for learning.

CASE 6: Sharing biscuits

(from Kazemi and Stipek, 2001)

Mathematics teaching for diverse learners:

- creates a space for the individual and the collective;
- provides opportunities for children to resolve cognitive conflict;
- involves the respectful exchange of ideas;
- involves explicit instructional discourse;
- provides opportunities for cognitive engagement and press for understanding.

Kazemi and Stipek (2001) observed the interactions within four teachers' lessons. They found that, despite the outward appearance in all classrooms of a focus on understanding, opportunities for students to participate in mathematical enquiry varied across the classrooms. Their study highlights those sociomathematical norms that promote students' engagement in conceptual mathematical thinking and conversations.

Targeted learning outcomes

The desired outcome of these tasks was for students to construct mathematical understanding about addition of fractions and to become skilled at communicating in mathematical language as they described and defended their differing mathematical interpretations and solutions.

Learning context

The study involved four teachers of grade 4 and 5, primarily low-decile classes, all teaching the same lesson on addition of fractions. The lesson involved the partitioning of brownie biscuits. The lesson plan provided one sample problem based on equivalence. Tools suggested for student use included sheets of paper with 16 pre-drawn squares.

On the surface, students appeared to be focused on understanding mathematics. However, a closer analysis of the interactions revealed differences in the way in which students were engaged in mathematical practices. In two of the four classrooms, the researchers categorised the interactions as consistently creating a high press for conceptual thinking. The other two classrooms were characterised as demanding a lower press for conceptual thinking.

The students were required to persuade each other by clear explanation and reasonable argument of their answer. In establishing rules of group participation, the teacher tried to ensure that the arguments for the errors were clearly voiced and justified.

Mathematical argumentation

This first vignette illustrates how students provided explanations that went beyond descriptions or summaries of the steps they used to solve the problem: they linked their problem-solving strategies to mathematical processes.

Luis: There were six crows, and we made, like, a colour dot on them ... There were four brownies, and we divided three of them into halves and the last one into sixths. One of the crows got $\frac{1}{2}$ and $\frac{1}{6}$.

Chris: Each crow got $\frac{1}{2}$ and $\frac{1}{6}$. In our second step, we had three brownies and we divided them in half. So each crow got $\frac{1}{2}$. $\frac{1}{2}$ plus $\frac{1}{6}$ equals $\frac{4}{6}$. So we have $\frac{1}{2}$ and $\frac{1}{6}$ and right here is $\frac{4}{6}$. [He points to two squares: one divided into half and then $\frac{3}{6}$ and the other into sixths. $\frac{4}{6}$ had been shaded in each brownie; see figure 7.6.]

Luis: Just to prove that it's the same. Then $\frac{4}{6}$ is what they got here, plus $\frac{1}{2}$. And $\frac{1}{2}$ is equal to $\frac{3}{6}$. $\frac{4}{6}$ plus $\frac{3}{6}$ is equal to one whole and $\frac{1}{6}$.

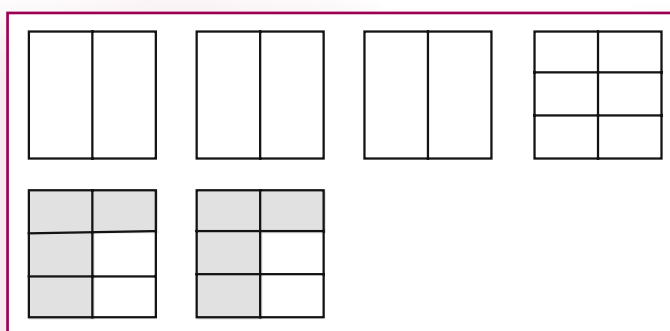


Fig. 7.6. Sharing 4 brownies among 6 crows to show that $\frac{1}{2}$ and $\frac{1}{6}$ equal $\frac{2}{6}$

It is clear from this episode that the students understood the need to demonstrate their mathematical argument for equivalence graphically as well as verbally. Kazemi and Stipek noted that the teacher invited everyone, not just the students at the board, to think about how the students had solved the problem. In the resulting discussion, the teacher required that students focus on the mathematical concept of equivalence and its relation to the process of adding fractional parts; it was not enough that students commented on the clarity of the drawings or how they were shaded.

To illustrate the effect of contrasting expectations, Kazemi and Stipek provide an episode from another classroom in which students engaged in the same social practices of describing their thinking but merely summarised the steps they took to solve a problem. The episode follows on from Raymond's description of his solution for dividing 12 brownies among eight people. The teacher, Ms. Andrew, had drawn 12 squares on the chalkboard. Raymond divided four of the brownies in half.

- Ms. A: Okay, now would you like to explain to us what ...
- R: Each one gets one, and I give them a half.
- Ms. A: So each person got how much?
- R: One and $\frac{1}{2}$.
- Ms. A: $\frac{1}{2}$?
- R: No, one and $\frac{1}{2}$.
- Ms. A: So you're saying that each one gets one and $\frac{1}{2}$. Does that make sense? [Chorus of "yeahs" from students; the teacher moves on to another problem.]

Ms. Andrew did not ask students to justify why they chose a particular partitioning strategy. A commonly observed practice in Ms. Andrew's class was for students to indicate with a show of hands or by calling out 'yes' or 'no', their responses to questions such as "How many people agree?" "Does this make sense?" or "Do you think that was a good answer?" These general responses, while on the surface indicating participation, revealed limited information about students' thinking or their understanding of the mathematical concepts involved.

Understanding relationships

Sharing solution strategies enables students to reflect on relationships within mathematics. The following vignette illustrates how one of the teachers in Kazemi and Stipek's study supported her students' examination of the mathematical similarities and differences among multiple strategies.

After Michelle and Sally had described their strategy and solution for a fair-sharing problem (see figure 7.7), the teacher turned to the class:

- Ms. M: Does anyone have any question about how they proceeded through the problem? ... What did they use or do that was different than what you might have done?
- Jeff: They used steps.
- Ms. M: Right, they divided it into steps. But there were some steps that I haven't seen anyone else use in the classroom yet.
- Carl: They added how many brownies there were altogether.
- Ms. M: Okay, so they used ...
- Jan: They divided into six, and there was one left over, and then they figured how they were going to divide that equally so that every crow gets a fair share.

Ms. M: Exactly, and that was very observant of you to see that. As I walked around yesterday, this is the only pair that used a division algorithm to determine that there was a whole brownie and a piece left over. So they did it in two different ways.

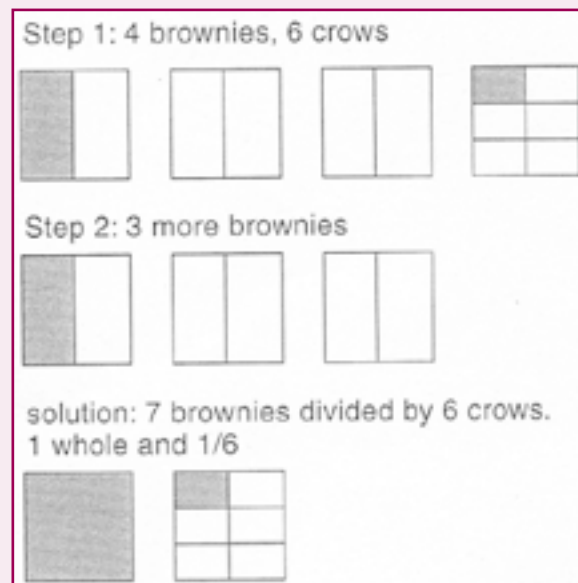


Fig. 7.7. Sharing 4 brownies among 6 cows and then 3 more brownies among 6 cows

In this exchange, the teacher asked her students to reflect on what was unique about a particular group's solution strategy. Students' responses included both organisational ("they divided it into steps") and mathematical ("they divided it into sixths") aspects. The researchers noted that the focus on mathematical differences among shared strategies supported students' formation of mathematical connections between various solution paths.

In contrast, in the 'low press for understanding' classrooms, discussions focused on non-mathematical aspects of shared strategies. The sharing of strategies "looked like a string of presentations, each one followed by applause and praise" (p. 72). Links, if they were made, consisted of non-mathematical aspects. For example, in Ms. Andrew's class, a pair of students reported that a solution strategy involved cutting the brownies and distributing the pieces to each individual. Another student reported drawing lines from the fractional parts of the brownies to the individuals who received them. Although both solutions used partitioning strategies, they were accepted as mathematically different, based on the way they handed out the pieces.

Building on errors

In classrooms where student participation and contribution is valued, student errors provide entry points for further mathematical discussion. Errors, or inadequate or partial solutions, provide opportunities to reconceptualise a problem, explore contradictions in a solution approach, or try out alternative strategies.

In the following episode, Ms. Carter organised a discussion based on conflicting student responses to the problem. By encouraging the whole class to think about a range of possible solutions, she created an opportunity for all students to engage in mathematical analysis. Following on from an earlier class presentation in which Sarah and Jasmine offer a diagrammatic solution to the problem of dividing nine brownies equally among eight people (fig. 7.8), Ms. Carter is involved in the following classroom exchange:

- Ms. C: Do you want to write that down at the top [of the OHP] so I can see what you did? [Jasmine writes $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$]
- Ms. C: Okay, so that's what you did. So how much was that in all?
- J: It equals $\frac{1}{8}$ or $\frac{6}{8}$
- Ms. C: So she says it can equal 6 [sic] and $\frac{6}{8}$?

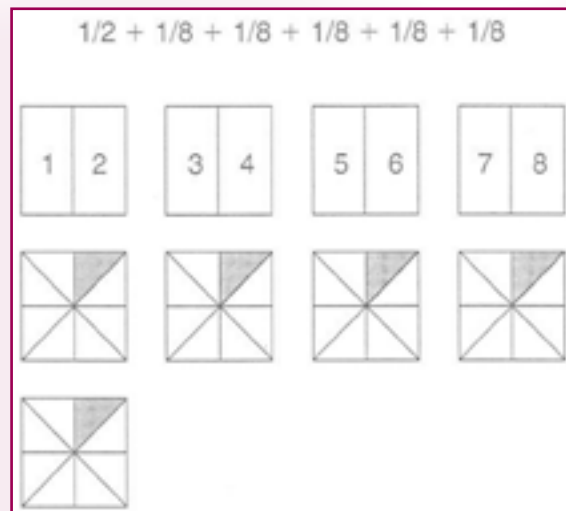


Fig. 7.8. Sharing 9 brownies among 8 people

In this episode, the teacher could have stepped in and pointed out why $\frac{6}{8}$ and $\frac{1}{8}$ are not equal. Instead, her response was to encourage her students to explore the error by explaining why $\frac{6}{8}$ and $\frac{1}{8}$ can't be equal.

Kazemi and Stipek also noted that Ms. Carter frequently used her observations of inadequate solutions during group work to plan whole-class instruction. In contrast, although incomplete or inadequate solutions were accepted as a normal part of learning in the other two classrooms, they were more likely to be ignored by the teacher or passed over until an adequate solution was offered or corrected by the teachers themselves.

Collaborative argumentation

In classes where there was a high press for understanding, collaborative work was accompanied by an expectation that each student was accountable for thinking through the mathematics involved in a problem. There was an expectation that consensus should be reached through mathematical argumentation. Teachers in both classrooms provided guidelines for small-group participation and reinforced these expectations. When Ms. Martin began the work on the fair-share problems, she made the following statement regarding individual accountability:

Teacher: Everyone in your group, whether it's just the two of you, or the three of you, everyone in your group needs to understand the process that you all were supposed to go through together. Because when you make a presentation, you don't know whether or not you are going to be asked a question. So you don't know if you're going to be asked by me or by your classmates. So you need to make sure that each person understands each part of the process you went through.

When working in groups, these expectations appeared to have been understood in terms of distribution of labour and contribution. The following episode is typical of an interactive discussion:

Keisha: See, this is how I explained it. [Reads.] "What we did is we took three brownies and cut them into half because three plus three equals six. And there are six crows."

Mark: This is what I put so far. [Reads.] "We knew that there were three more brownies, and we divided each one in half. One brownie had two halves, and another brownie had two halves, and another one had two halves."

Keisha: Just write, "And all three had two halves." And all ...

Mark: [Starts writing.] "And all three ..." I don't have to write.

Keisha: Okay, and each crow got $\frac{1}{2}$ and ... just write: " $3 + 3$ is 6 " so each crow got $\frac{1}{2}$ of the brownie.

Mark: Yeah, but these are halves. [Counts halves in each brownie.] 2, 2.

Keisha: Yeah, I know. 1, 2, 3. [Counts brownies.] Cut them in half, 1, 2, 3, 4, 5, 6. [Counts halves.] There were six crows. Each crow got $\frac{1}{2}$.

Mark: [Writes "Each crow got one-half."]

Having written about one of the steps in the problem, Mark and Keisha proceeded to evaluate and expand their written explanations. We see in the above interaction how Mark indicated that he did not appear to understand how " $3 + 3 = 6$ " applied. Keisha's explanation that she was referring not to the halves themselves,

but that the number of halves corresponded to the number of crows appears to have facilitated a mutual understanding.

In contrast to the high-press classrooms, the teachers in the low-press classrooms only gave general instructions (such as “work with a partner” or “remember to work together”) to support collaboration. The researchers noted frequent occurrences of unequal distribution of work within groups. Students who were unclear about what to do often withdrew and allowed another student to take over. In the following example, Ellen was excluded from participation. Without an opportunity to think about the problem, her role was one of ‘listen and agree’:

- Lisa: We need five brownies. So see, 1, 2, 3, 4. So we cut these into half. So 1 ... 8. They get a half each. And then there's one more cookie and eight people, so we just cut into eighths, and it'll be even for everybody.
- Ellen: Wow, you did that fast. I didn't even do anything.
- Lisa: I knew there's 5, and I knew 4; 2 times 4 is 8.
- Ellen: Oh, I get it.

Learner outcomes

Students in classrooms with established social and sociomathematical norms for mathematical thinking were more likely to be observed engaging in mathematical discussion that was conceptual rather than procedural in nature.

Quality pedagogy

The difference between the high- and low-press exchanges illustrates that, to support conceptual thinking, teaching needs to go beyond superficial practices based on the discussion and sharing of strategies. Kazemi and Stipek argue that pedagogical practices need to be based on the following sociomathematical norms that work together to establish expectations about what constitutes mathematical thinking:

- An explanation consists of a mathematical argument, not simply a procedural description or summary.
- Mathematical thinking involves understanding relations among multiple solutions.
- Errors provide opportunities to reconceptualise a problem, explore contradiction in solutions, or pursue alternative strategies.
- Collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

Belonging to a community of learners

Engaging students in what Wood, Cobb, & Yackel (1991) call “genuine conversations” about mathematics means that teachers take students’ ideas seriously in their attempts to support students’ understanding. CASE 7 provides a window into a lesson that involves ratios. In addition to featuring the use of multiple connections between percentage, fractions, and measurement, the detailed exchange between the teacher and student illuminates the sense of community and ethic of care that pervade this classroom.

CASE 7: Mixing Drinks

(from Sherin, Mendez, and Louis, 2004)

Mathematics teaching for diverse learners:

- demands an ethic of care;
- involves the respectful exchange of ideas;
- creates a space for the individual and the collective;
- provides opportunities for students to resolve cognitive conflict;
- involves sequencing of tasks and provision of appropriate challenge;
- provides opportunities for students to problematise activities based in realistic contexts;

- involves explicit instructional discourse.

This case is taken from a wider curriculum reform programme, *Fostering a Community of Learners* (FCL; Brown & Campione, 1996). The goal in the mathematics classroom was the building of a discourse community that focused on students' explanations and discussion of their ideas. In the following example, we see how the FCL principles of activity, reflection, collaboration, and community are realised in David's classroom.

Targeted learning outcomes

Solving ratio problems.

Learning context

The students in this study were from a middle-school classroom involved in an extended reform programme intervention in conjunction with researchers at Stanford University. Data were collected through videotapes of instruction and through discussions of these videotapes with the teacher. The lesson reported on in this case occurred in the second year of the study. Students were given a series of drink mix problems—one of these was as follows:

Juice Mix A contains 2 cups of concentrate and 3 cups of cold water. Assuming that each camper will get $\frac{1}{2}$ cup of juice, how much concentrate and how much water are needed to make juice for 240 campers?

While the students worked on the problem in groups, David, the teacher, circulated through the class.

Student and teacher activity

As David approached Antoine's group, Antoine called him over for help on the last problem. David asked Antoine what he had done so far.

- Antoine: OK. You do 3 out of 5. Three divided by 5 is 60%, times 240 equals 144, divided by 2 is 72. I've got the answer. I've got skills, boy. Yeah.
- Teacher: Can you explain what you just did, what that means?
- Antoine: Yeah, yeah.
- Teacher: What's the 3 out of 5 part?
- Antoine: Three out of 5 is the number, the number, the cups of water divided by all of the cups put together. And then it equals 60%. And then, times 240 is the number of campers.

Antoine had recognised that for every five cups of juice, three of the cups were water. And it seemed that because $\frac{3}{5} = 0.6$, Antoine concluded that 60% of the juice mix must be water, no matter how many cups of juice. He then took the total number of campers, 240, and multiplied that by 0.6. Because each camper gets only half a cup of juice, Antoine divided 144 by 2 to get 72 cups of water in the total mix. Later on in the discussion, it became clear that Antoine was unsure of this last part of his calculation.

The report of the teacher's reflections of this episode noted that he was unclear as to why Antoine would multiply 0.6 by 240—the number of campers. Why did Antoine need to know how many 60% of the campers would be? David's own solution involved first calculating the total numbers of cups of juice that were needed, 120 cups and then because 60% of the juice was water, calculating 60% of 120 to conclude that there 72 cups of water. David asked Antoine to elaborate his solution method.

- Teacher: So that tells you what, if you do 60% times 240?
- Antoine: It tells you how many cups, wait. Times 240. Tells you how many cups are needed.
- Teacher: That tells you 60% of the campers.
- Antoine: No, tells you 60% of that juice stuff.
- Robert: Tells you that 60% of the mix is concentrate.
- Antoine: No, it tells you that 60% of water is in the mix altogether.
- Robert: Yeah.
- Teacher: All right, all right, you've got skills. Let's go.
- Antoine: I know I've got skills, you ain't got to tell me. Then this is the part I messed up at. The number of campers, you get. You have to times it, and then what do you get?

In the above episode, we see that Antoine's explanation initially involves 'how many cups' are needed. In response to the teacher probe Antoine becomes a little more precise, explaining that it tells him '60% of

that juice stuff'. Robert, another group member, interjects with an incorrect statement, saying that it tells you that '60% of the mix is concentrate'. However, in trying to respond to Robert, Antoine is finally able to explain that 60% of 240 tells him how much 'water is in the mix altogether'.

Quality pedagogy

The researchers analysed factors that facilitated students' success with the ratio problems in terms of four principles of learning: activity, reflection, collaboration, and community.

- Antoine is clearly an active participant in the discussion. He seeks teacher assistance and enthusiastically reviews his solution method with the teacher.
 - The interaction supports Antoine to "reflectively turn around on [his] own thought and action and analyse how and why [his] own thinking achieved certain ends or failed to achieve others" (Shulman, 1995, p. 12, cited in Sherin et al., 2004). In reviewing his solution, Antoine clearly wants to resolve his uncertainty as to why he decided to divide 144 by 2.
 - Collaboration involves teacher–student and student–student interactions. In the second episode, Robert and Antoine scaffold and support each other's learning in ways that supplement each other's knowledge.
 - Sherin, Mendez, and Louis claim that the 'community' principle is more clearly evident in the videotape. From the transcripts, it is apparent that the teacher and Antoine have an established routine that supports effective communication. Teacher questioning occurs in an environment in which Antoine feels safe to respond. Antoine seeks teacher help, knowing that it is appropriate to question his own solution despite the fact that he already has the answer. The banter between Antoine and the teacher provides evidence of a culture that affords opportunities for students to share their understandings and to know that their opinions are valued.
-

Teacher knowledge: Forms of 'knowing' fractions

In all of the CASEs, the centrality of teacher knowledge is evident. Reiterating discussions from earlier chapters, what teachers do is very dependent on what they understand about the teaching and learning of mathematics. For fractions, in particular, knowing different models and various approaches to the teaching of fractions places high demands on teachers' mathematical and pedagogical content knowledge.

Unfortunately, numerous studies point to shortcomings in teachers' understanding of rational numbers (e.g., Domoney, 2001). In a seminal study comparing US and Chinese teachers' mathematical knowledge, Ma (1999) demonstrated how US teachers more readily situated fraction problems in real-world contexts. However, this apparent familiarity with and link to everyday experience appeared to be superficial. Ma found substantial differences in knowledge when teachers were asked to perform division of fractions or to generate representations of fractions. For example, only one of the 23 US teachers in the study generated a conceptually correct representation for the meaning of the equation $1\frac{3}{4} \div \frac{1}{2}$. This compares with 65 of the 72 Chinese teachers. Among the 23 US teachers, 6 could not create a story to match the calculation and 16 provided stories that contained misconceptions. Twelve of the misconceptions involved confusing division by $\frac{1}{2}$ with division by 2 or multiplication by $\frac{1}{2}$. For example: *Jose has one and three-fourths boxes of crayons and he wants to divide them between two people or divide the crayons in half, and then, first we could do it with crayons and maybe write it on the board or have them do it in numbers.* Ma cautioned that although US teachers reported the frequent use of real contexts, the 'real world' cannot produce the mathematical content by itself. She claims that "without a solid knowledge of what to present, no matter how rich one's knowledge of students' lives, no matter how much one is motivated to connect mathematics with students lives, [this is without benefit] if one still cannot produce a conceptually correct representation" (p. 82).

CASE 8: Representation of Division by Fractions

(from Ma, 1999)

All of the Chinese teachers successfully computed $1\frac{1}{2} \div \frac{1}{2}$ and 65 of the 72 created a total of more than 80 story problems representing the meaning of division by a fraction. The Chinese teachers represented the concept using three different models of division: measurement (quotitive), partitive (sharing), and product and factors. For example, $1\frac{3}{4} \div \frac{1}{2}$ might represent:

- $1\frac{3}{4}$ metres $\div \frac{1}{2}$ metre = $7\frac{1}{2}$ (quotitive model)
- $1\frac{3}{4}$ metres $\div \frac{1}{2} = 7\frac{1}{2}$ metres (partitive model)
- $1\frac{3}{4}$ square metre $\div \frac{1}{2}$ metre = $7\frac{1}{2}$ metres (product and factors)

corresponding to the problems:

- How many $\frac{1}{2}$ m lengths of timber are there in $1\frac{3}{4}$ m of timber?
- If half a length of timber is $1\frac{3}{4}$ m, how long is the whole piece of timber?
- If one side of a $1\frac{3}{4}$ square metre rectangle is $\frac{1}{2}$ m, how long is the other side?

In their discussions of the meaning of division by fractions, the Chinese teachers mentioned several concepts that they considered related to the topic. These are represented in the diagram:

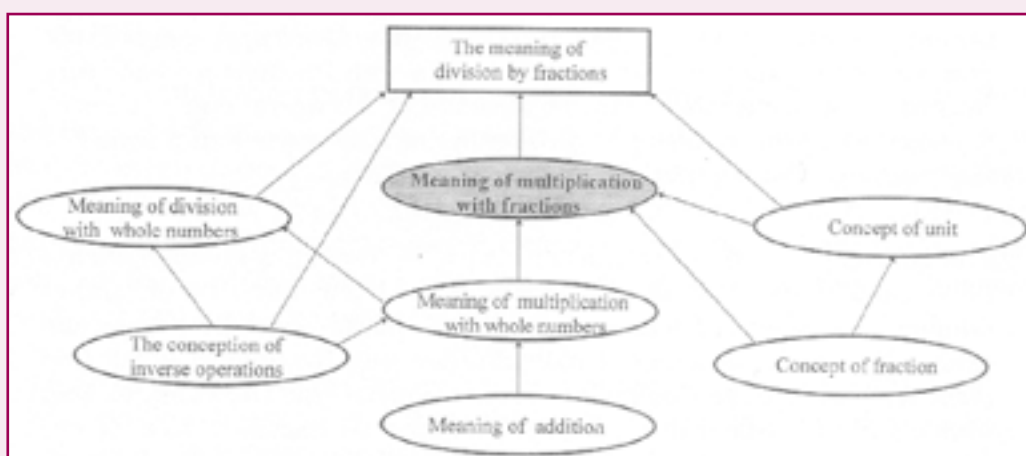


Fig. 7.9. Connected knowledge for understanding the meaning of division

Their view of connected knowledge translated into a forward trajectory of learning. Work on division by fractions was also valued for the role in intensifying concepts of rational number already encountered by the students. The Chinese teachers expressed the view that students may “gain new insight through reviewing old ones [concepts]. The current learning is supported by, but also deepens, the previous learning” (p. 77).

A more recent comparative study of US and Chinese teachers (Shuhua, Kulm, & Wu, 2004) also notes the different system demands on teachers’ pedagogical content knowledge. The researchers express concern about the pedagogical approaches in the US system that indicate a “lack of connection between manipulative and abstract thinking, and between understanding and procedural development” (p. 170).

As we have seen earlier, teacher knowledge is also a significant factor in the interpretation of students’ thinking and solution strategies. For example, in assessing students’ understanding of fractions, sound pedagogical content knowledge is needed to determine the effectiveness of different tasks.

CASE 9: Knowledge of Students Solving Fractions

(from Grossman, Schoenfeld, and Lee, 2005)

Which of the following tasks would best assess whether a student can correctly compare fractions?

- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}$, $\frac{1}{4}$, $\frac{11}{16}$
- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}$, $\frac{3}{4}$, $\frac{1}{16}$
- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}$, $\frac{3}{4}$, $\frac{11}{16}$

Can you explain why the two tasks you did not select are not good assessments of students' understanding of fractions?

To do this, teachers need to know the relevant mathematics in a deep and connected way (Ma, 1999). Teachers need more than just the ability to solve the problem. They also need to know about the ways in which students might solve the problem and the reasoning that they may or may not use. For example, there are at least two ways to solve the problem: converting the fractions to decimals and comparing them or reasoning through the task by comparing the fractions themselves. Of the three fractions in the first set, only one, $\frac{1}{4}$, is less than $\frac{1}{2}$. So $\frac{1}{4}$ is the smallest. And then because $\frac{5}{8} = \frac{10}{16}$, and $\frac{10}{16}$ is less than $\frac{11}{16}$, $\frac{5}{8}$ is less than $\frac{11}{16}$. Thus the order is $\frac{1}{4}$, $\frac{5}{8}$, $\frac{11}{16}$. But the real issue for the teacher is how their own students will solve this problem.

Research shows that many students will focus only on the number of pieces, not their relative size. For example, given the first set of three fractions, students will think, " $\frac{1}{4}$ has only one piece, so it's the smallest; $\frac{5}{8}$ has five pieces, so it's in the middle; and $\frac{11}{16}$ has eleven pieces, so it's the largest." Unfortunately this incorrect reasoning produces the right answer. Another common misconception held by students is that "the smaller the pieces, the smaller the fraction." Because sixteenths are smaller than eighths and eighths are smaller than fourths, students using this reasoning may arrive at the correct answer, given the second set of three fractions. In the case of the third set of three fractions, however, students who use either of these forms of incorrect reasoning will get the wrong answer—and their wrong answer will suggest why they got it wrong.

With fractions, as with other areas of mathematics, teachers need to distinguish what the student understands as opposed to what the student can do (Pearn & Stephens, 2004). The teacher may need to listen across multiple tasks in order to determine how a student is thinking. The following response by Madison, a student in Mitchell and Clarke's (2004) research, illustrates the value of multiple tasks in the diagnostic setting. When Madison was asked to respond to an estimation task involving adding pairs of fractions near 1 and near $\frac{1}{2}$, her estimate for $\frac{7}{8} + \frac{12}{13}$ was "two"—the correct answer. According to the researchers, her reasoning appeared modest but faultless: "I just guessed. That's seven bits of eight. Twelve bits of thirteen." Based on the assumption that Madison had used a part-whole approach, her follow-up comment seemed strange and unrelated to her answer, "And I just added eight and thirteen". Madison was then asked to estimate the answer to $\frac{3}{8} + \frac{5}{12}$, to see if she could use half as a benchmark. Her answer "twelve" was accompanied with the following reasoning: "But eight and twelve are twenty and the three and the five are covering a bit of it and so I took it away." It appeared that adding the numerators and taking away that from the sum of the denominators was her procedure. Applying this whole-number procedure to the previous question, it is clear how she arrived at the answer of two; $8 + 13$, which is what she said she did, is 21. $21 - (7 + 12)$ is 2.

These examples of teachers' knowledge about fractions were sourced from research studies that *interviewed* teachers. They serve to reinforce the evidence that we have presented throughout the synthesis, which clearly illustrates the critical relationship between teacher knowledge enacted in the learning environment and learner outcomes. Specifically, teachers who have strong pedagogical content knowledge and 'connected' knowledge of mathematics, and who also display an awareness of the need to 'connect' with learners' understandings of mathematics, are most likely to occasion quality learning opportunities for diverse learners.

Part–whole pedagogy

Individually, these CASEs illuminate those aspects of effective pedagogy for diverse learners that have been discussed in earlier chapters. As a whole, they represent more than the parts of the story. They illustrate the truth that effective pedagogy cannot be captured by a list of ticks in boxes. Teaching is socially situated and contextually bound. What works well with one group of students may not work so well with another, and what works well with one mathematical topic may not work so well with another. The CASEs serve to illustrate how students and teachers are collectively learners, constructing meanings that contribute to the overall success of each learning experience. In each CASE, there is a commitment to the fundamental principle proffered in chapter 2, and that is that all students, irrespective of age, have the capacity to become powerful learners of mathematics.

¹ This problem is also discussed in the context of teacher knowledge in Schifter, D. (2001). Learning to see the invisible. In T. Wood, B. Scott-Nelson & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109–134). Mahwah, NJ: Lawrence Erlbaum Associates.

References

- Angier, C., & Povey, H. (1999). One teacher and a class of school students: Their perception of the culture of their mathematics classroom and its construction. *Educational Review*, 51, 147–160.
- Askew, M. (in press). Section 3 Introduction. In M. Brown, M. Askew & A. Millet (Eds.), *Learning about number: Interactions and outcomes in primary classrooms*. Netherlands: Kluwer Academic Publishers.
- Brown, A., & Campione, J. (1996). Psychological theory and the design of innovative learning environments. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 188–228). New York: Cambridge University Press.
- Christensen, I. (2004). *An evaluation of Te Poutama Tau 2003*. Wellington: Ministry of Education.
- Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & C. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's developments*. Oxford: Oxford University Press.
- Davis, G., Hunting, R., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior*, 12, 63–76.
- Domoney, B. (2001). Student teachers' understanding of rational numbers. In J. Winter (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 21). BSRLM.
- Empson, S. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, 17(3), 283–342.
- Empson, S. (2003). Low performing students and teaching fractions for understanding: An interactional analysis. *Journal for Research in Mathematics Education*, 34(4), 305–343.
- Grossman, P., Schoenfeld, A., & Lee, C. (2005). Teaching subject matter. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world* (pp. 201–231). San Francisco: Jossey-Bass.
- Hart, K. (1988). Ratio and proportion. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 198–219). Reston: NCTM.
- Henningsen, M., & Stein, M. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524–549.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Higgins, J. (2005). Equipment-in-use in the Numeracy Development Project: Its importance to the introduction of mathematical ideas. In J. Higgins, K. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 89–96). Wellington: Ministry of Education.
- Houssart, J. (2002). Simplification and repetition of mathematical tasks: a recipe for success or failure? *The Journal of Mathematical Behavior*, 21(2), 191–202.
- Hunting, R., & Sharpley, C. (1988). Fraction knowledge of preschool children. *Journal for Research in Mathematics Education*, 19(2), 175–180.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal*, 102(1), 59–80.

- Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Hillsdale: Lawrence Erlbaum Associates.
- Kieren, T., Davis, B., & Mason, R. (1996). Fraction flags: Learning from children to help children learn. *Mathematics Teaching in the Middle School*, 2(1), 14-19.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.
- Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah NJ: Lawrence Erlbaum Associates.
- Mitchell, A., & Clarke, D. (2004). When is three quarters not three quarters? Listening for conceptual understanding in children's explanations in a fraction interview. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 367-373). Sydney: MERGA.
- O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, 46, 143-185.
- Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what Year 8 students really think about fractions. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 430-437). Sydney: MERGA.
- Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 13, 307-317.
- Sarage, L. (1992). Becoming a mathematical thinker. In D. Schifter & C. Twomey-Fosnot (Eds.), *Reconstructing mathematics education* (pp. 63-82). New York: Teachers College Press.
- Sharp, J., Garofalo, J., & Adams, B. (2002). Children's development of meaningful fraction algorithms: A kid's cookies and a puppy's pills. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios and proportions* (pp. 18-28). Reston: National Council of Teachers of Mathematics.
- Sherin, M. G., Prentice, M., & Louis, D. (2004). A discipline apart: the challenges of "Fostering a Community of Learners" in a mathematics classroom. *Journal of Curriculum Studies*, 36(2), 207-232.
- Shuhau, A., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school mathematics teachers in China and the US *Journal of Mathematics Teacher Education*, 7, 145-172.
- Sinicrope, R., Mick, H., & Kolb, J. (2002). Interpretations of fraction division. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 153-161). Reston: National Council of Teachers of Mathematics.
- Smith, C. (1998). A discourse on discourse: Wrestling with teaching rational equations. *The Mathematics Teacher*, 91(9), 749-753.
- Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33(2), 455-488.
- Thompson, P., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, L. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 95-113). Reston: National Council of Teachers of Mathematics.
- Turner, J., Meyer, D., Cox, K., Logan, C., DiCintio, M., & Thomas, C. (1998). Creating contexts for involvement in mathematics. *Journal of Educational Psychology*, 90(4), 730-745.
- Watson, A., & De Geest, E. (2005). Principled teaching for deep progress: Improving mathematical learning beyond methods and material. *Educational Studies in Mathematics*, 58, 209-234.
- Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, 28, 6.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 2(4), 458-477.
- Young-Loveridge, J. (2005). Patterns of performance and progress: Analysis of 2004 data. In J. Higgins, K. C. Irwin, G. Thomas, T. Trinick & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 5-20). Wellington: Ministry of Education.

8. Endings Marking New Beginnings

This Best Evidence Synthesis has examined the links between pedagogical practice and student outcomes. We have presented and interpreted instances of deliberate and effective efforts by teachers to do the best possible job for their students. Our synthesis sketches out in broad strokes evidence of their quality teaching. As in the other syntheses undertaken as part of the Iterative Best Evidence Syntheses Programme, this evidence draws on the histories, cultures, language, and practices found in New Zealand and in comparable international contexts. What is distinctive about the approach taken by the New Zealand Iterative BES programme is its consideration of “all forms of research evidence regardless of methodological paradigms and ideological rectitude, and its concern in finding contextually effective, appropriate and locally powerful examples of ‘what works’” (Luke & Hogan, 2006). Its commitment is to the development of knowledge and pedagogical competencies suited for our bicultural society, for our cosmopolitan citizenship, and for educational change.

We have conceptualised mathematics teaching as part of a nested system. Teaching is not so much a discrete entity, but a unity nested within other unities such as the classroom, the school, the family, and the wider school community. In this conceptualisation, teaching is influenced by adaptive rather than additive factors and by interactive rather than isolated variables. This means that the outcomes of teaching are contingent on a network of interrelated factors and environments. They are informed by teachers’ active engagement with processes and people, institutional settings, and home practices. These are the factors and conditions that shape how, and with what effect, mathematics is taught and learned. Within the nested system, teachers modify and transform their pedagogical practice: they adapt their practice in relation to their personal understandings, the system-level processes of the school and other educational institutions, and the understandings and support of families and the wider school community.

Several important ideas follow from this. First, teaching is a complex activity. Quality teaching is not simply the fact of ‘knowing your subject’ or the condition of ‘being born a teacher’. Second, by nesting teaching within a systems network, we cannot claim that teaching causes student outcomes. Understanding this prevents us from romanticising mathematics teaching; it also prevents us from reducing effective practice to prescriptive decree. But if student outcomes are not *caused* by teaching practices, they can at least be *occasioned* by those practices. In this synthesis, we have offered important insights from research about how that occasioning might take place. Certain patterns have emerged that have enabled us to foreground ways of doing and being that mark out an effective pedagogical practice. Each aspect, of course, constitutes but one piece of evidence and must be read as accounting for one variable, amongst many, within the teaching nested system. Taking all these aspects together allows us to envisage what quality teaching might look like.

Our synthesis has deepened our understanding of mathematics teaching in many ways, and we attempt here to summarise the insights we have gained. We have found that building home–community and school–centre partnerships is fundamental to effective teaching. Facilitating harmonious interactions between school, family, and community contributes to the enhancing of students’ aspirations, attitudes, and achievement. Teachers who collaborate with parents, families/whānau, and community members come to understand their students better. Parents benefit too. Through their purposeful involvement in centre/school activities, by assisting with homework, and in providing suitable games, music, and books, they develop a greater understanding of the centre/school’s programme and an appreciation of their children’s mathematical knowledge. The home and community environments offer a rich source of mathematical experiences on which to scaffold centre/school learning.

Early childhood centres offer most children their first experience of a formal educational institution and provide an environment where powerful mathematical ideas can be developed. Researchers have found that effective teachers provide opportunities for children to explore

mathematics through a range of imaginative and real-world learning contexts. The most effective settings provide a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities. Children's play/exploration is an important site of mathematical learning. Informal mathematical knowledge typically arises out of children's everyday activities: counting, playing with mathematical shapes, telling time, estimating distance, sharing, cooking, and playing games. Quality teaching builds on this informal knowledge, capturing the learning opportunities within the child's environment and making available a range of resources and activities. Contexts that are rich in perceptual and social experiences support the development of problem-solving and creative-thinking skills.

Within centres and classrooms, teachers facilitate learning for diverse learners by truly caring about student engagement. Research in this area has found that effective teachers demonstrate their caring by establishing learning spaces that are hospitable as well as academically 'charged'. They work at developing interrelationships that create spaces for students to develop their mathematical and cultural identities. Teachers who care work hard to find out what helps and what hinders students' learning. They have high yet realistic expectations about enhancing students' capacity to think, reason, communicate, reflect upon, and critique their own practice, and they provide students with opportunities to ask why the class is doing certain things and with what effect. At the same time, research quite clearly demonstrates that pedagogy focused solely on the development of a trusting climate and on listening to students' ideas does not get to the heart of what mathematics teaching truly entails. Classroom work is made more enriching when discussion involves co-construction of mathematical knowledge through the respectful exchange of ideas. When teachers work at developing inclusive partnerships, they ensure that the ideas put forward are, or become, commensurate with mathematical convention and curricular goals.

A context that supports students' growing awareness of themselves as legitimate participants in the production of mathematical knowledge creates a space for both the individual and the collective. Many researchers have shown that small-group work can provide the context for social and cognitive engagement, while others have cautioned that limited-English-speaking students are less inclined to share their thinking in group process. Some students, more than others, appear to thrive in class discussion groups. A personal reluctance to participate and the low social obligations and cognitive demands unwittingly placed by teachers on some students have the effect of excluding them from full engagement in mathematics. Within the classroom, all students need to participate. And they all need time alone, away from the demands of a group, to think and work quietly. This line of research has also revealed that classroom grouping by ability has its problems as a pedagogical practice. Teachers who teach lower streamed classes tend to follow a protracted curriculum and offer less varied teaching strategies. This organisational practice has a detrimental effect on the development of a mathematical disposition and on students' sense of their own mathematical identity.

Effective teaching for diverse students demands teacher knowledge. Studies exploring the impact of content and pedagogical knowledge have shown that what teachers do in classrooms is very much dependent on what they know and believe about mathematics and on what they understand about the teaching and learning of mathematics. Successful teaching of mathematics requires a teacher to have both the *intention* and the *effect* to assist pupils to make sense of mathematical topics. A teacher with the intention of developing student understanding will not necessarily produce the desired effect. Unless teachers make good sense of the mathematical ideas, they will not develop the flexibility they need for spotting the golden opportunities and wise points of entry that they can use for moving students towards more sophisticated and mathematically grounded understandings. Sound teacher knowledge is a prerequisite for accessing students' conceptual understandings and for deciding where those understandings might be heading. It is also critical for accessing and adapting resources to bring the mathematics to the fore. There is now a wealth of evidence available that shows how teachers' knowledge can be developed

with the support and encouragement of a professional community of learners.

Studies have provided conclusive evidence that teaching that is effective is able to bridge students' intuitive understandings and the mathematical understandings sanctioned by the world at large. Consistently emphasised in research is the fact that teaching is a process involving analysis, critical thinking, and problem solving. Language, of course, plays a central role. The teacher who has the interests of learners at heart ensures that the home language of students in multilingual classroom environments connects with the underlying meaning of mathematical concepts and technical terms. Teachers who make a difference are focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics. The responsibility for the distinguishing between terms and phrases and sensitising their particular nuances weighs heavily with the teacher, who profoundly influences the mathematical meanings made by students in the class.

Effective teaching at all levels ensures that mathematical tasks are not simply 'fillers' but require the solving of genuine mathematical problems. For all students, the 'what' that they do is integral to their learning. The 'what' is the result of sustained integration of planned and spontaneous learning opportunities made available by the teacher. Planning is done with many factors in mind, some determined by the individual student's knowledge and experiences, and others mediated by the pedagogical affordances and constraints and the participation norms of the classroom. Extensive research in this area has found that effective teachers use tasks that allow students to access important mathematical concepts and relationships, to investigate mathematical structure, and to use techniques and notations appropriately and that they employ such tasks over sustained periods of time. Tasks of this kind provide students with opportunities for success, present an appropriate level of challenge, increase students' sense of control, and develop valuable mathematical dispositions. In short, quality teaching enables students to develop habits of mind whereby they can engage with mathematics productively and make use of appropriate tools to support their understanding.

Gaps and omissions

We note that recent research into mathematics education now centres on the teacher and looks into the process of, and interrelationships within, teaching practice. Today's research tends to emphasise the co-construction of knowledge. Hence it explores the social context of teaching and learning through mostly qualitative research designs.

We make these observations alongside the caveat that we do not claim to have covered the field completely and absolutely. In offering this synthesis, we are mindful that there are gaps. Some omissions have arisen because it has not been possible to include every piece of research evidence about quality teaching. Given the specific New Zealand focus of the synthesis, and the breadth of available international work, only a sample of work undertaken beyond New Zealand could be included. We acknowledge, too, that a number of major New Zealand investigations of international repute have not been included because they focus on other areas of mathematics education.

Other gaps have arisen simply because gaps exist in the literature. To date, we do not know much about effective teaching in New Zealand at the secondary school level and we do not know how quality teaching might be characterised for Pasifika students. We might also note that, in common with research interests in Australia, very few researchers in New Zealand are exploring mathematics in early childhood centres or the impact of technology on mathematics learning and pedagogy.

The New Zealand literature also lacks longitudinal, large-scale studies of teaching and learning, both studies undertaken in New Zealand and studies undertaken in collaboration with overseas researchers. Such research is crucially important for understanding teachers' work and the impact of curricular change. The scholarly exchange of ideas made possible through joint projects with the international research community contributes, in numerous

ways, to the capability of our local researchers. As a consequence, the New Zealand research evidence in this synthesis—apart from the data available for the NDP—is mainly drawn from smaller projects and the postgraduate investigations undertaken by emerging mathematics education researchers.

Final words

This evidence-based view of quality centre and classroom mathematics pedagogy provides a springboard and a language for discussing reform, innovation and change in mathematics pedagogy. Through it we have developed new knowledge about effective mathematics teaching. We have learned, from the practices that enhance students' learning, about how schooling and non-schooling factors interact with each other in exerting influence on student outcomes. When investigating 'what works' we have learned a lot about 'how' and 'why' it works. We are privileged to share this knowledge with our readers.

What has been highlighted for us is the enormous complexity of teaching. Both experienced and new teachers, as well as other educators, often fail to appreciate the full span of pedagogical practice and how lesson plans might build on the intentions and the individual and collective histories and competencies of their students. Our hope is that this synthesis not only makes that complexity explicit but that it also provides ways of doing things that will assist teachers in their endeavours.

To innovate, reform, and take this synthesis to the next level will require all concerned people, from students to policy makers, to make an effort to keep the conversation going. Negotiating and making decisions in classes, in school mathematics teams, in teacher education programmes, and in publications will enable us all to come to view mathematics teaching in new ways. Everyone involved in schooling—teachers, principals, teacher educators, learners, specialist support services, boards of trustees, and policy makers—has a role to play in applying the evidence. Applying the evidence requires that we keep in sight our primary objective of producing high-quality learning outcomes for all our students. In turn, a pedagogical practice that prioritises the learner will generate new kinds of evidence about quality teaching.

There is a lot that all of us can learn from each other about quality pedagogical practices. There is no more important responsibility sitting with a school, a community, or a nation than to do the best job that it can in providing teachers with the knowledge and the incentive to facilitate learning for all students. The Ministry of Education has taken this responsibility seriously and acknowledges the role research can play as a form of discourse leading to pedagogical change. This synthesis of research-based evidence concerning quality teaching, commissioned by the Ministry, is one attempt to enhance the mathematics teaching in our centres and schools.

Reference

Luke, A., & Hogan, D. (2006). Redesigning what counts as evidence in educational policy: The Singapore model. In J. Ozga, T. Popkewitz, & T. Seddon (Eds), *Education research and policy: Steering the knowledge-based economy. World handbook of education* (pp. 170-184). London: Routledge.

Appendix 1: Locating and Assembling BES Data

Using the ‘health-of-the-system’ approach, we sought to examine the various factors implicated in the creation of an effective learning community. We investigated a number of measures that fell naturally from the ‘what’, ‘why’, ‘how’, and ‘under what conditions’ questions concerning pedagogical approaches that facilitate learning for all students. The task was a considerable one, involving information management, the engagement of advisory and audit groups, and the seeking of contributions from the education community in general and the mathematics education community in particular. This level of engagement ensured that the Best Evidence Synthesis would be inclusive of views from across the community.

Our initial search strategy required us to pay attention to different contexts, different communities, and multiple ways of thinking and working. With this in mind, we undertook a literature search that crossed national and international boundaries. We used a range of search engines as well as personal networks to help us find academic journals, theses, projects, and other scholarly work with a focus on mathematics in New Zealand schools and centres, and by selected authors worldwide. We searched both print indices and electronic indices, endeavouring to make our search as broad as possible within the limits of manageability. This search took into account relevant publications from the general education literature and from the literature that relates to specialist areas of education. The search covered:

- key mathematics education literature including all major mathematics education journals (e.g., *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *Journal of Mathematics Teacher Education*, *For the Learning of Mathematics*, *The Journal of Mathematical Behaviour*), international conference proceedings (e.g., PME, ICME), Mathematics Research Group of Australasia publications, and international handbooks of mathematics education (e.g., Bishop et al., 2003);
- relevant New Zealand-based studies, reports, and thesis databases, supported by input from the professional community and the Ministry of Education;
- education journals (e.g., *American Educational Research Journal*, *British Educational Research Journal*, *Cognition and Instruction*, *The Elementary School Journal*, *Learning and Instruction*, etc.) and New Zealand work (e.g., SAMEpapers, SET, NZJES);
- specialist journals and projects, especially those located within the wider education field (e.g., *New Zealand Research in Early Childhood Education*; *Journal of Learning Disabilities*);
- landmark international studies including TIMSS, PISA, the UK Leverhulme projects.

This search strategy led us to a large body of literature that had something to say about facilitating mathematics learning: the total number of sourced references was just over 1100. Table 1 categorises these references by source:

Table 1: Initial Database Composition by Source

Source of data	Relative frequency (n ~1100)
Mathematics education journals	24%
Mathematics education reports, books, handbooks	16%
Mathematics education conference proceedings	15%
Theses and projects	6%
General education reports, books, handbooks	10%
General education journals, reports, and proceedings	19%
Specialist journals	10%

All entries were stored and categorised using EndNote. To assist in the initial synthesis, we distinguished between ‘research’ and ‘discussion document’, and categorised entries according to (a) our ‘diversity’ descriptors (e.g., ethnicity, gender, socioeconomic), (b) centre/school level, and (c) country-of-origin of the data.

These categories and sub-divisions served as a useful starting point for overviewing the literature and allowed us to foreground our fundamental intent to be responsive to diversity. In addition, by classifying entries according to sector and country of origin, we gave ourselves a constant reminder of the need to be inclusive of all perspectives and interests. This inclusiveness gave us a body of literature comprising diverse frameworks and eclectic methodological and analytic approaches.

Selecting the evidence

Given the complexity of the teaching and learning process, it is not an easy matter to link specific outcomes with specific pedagogical approaches. In our first pass through the literature, we noted that studies could claim that student achievement was *influenced* by pedagogical practice much more readily than they could explain *how* that practice affected student achievement. Many studies offered detailed explanations of student outcomes yet failed to draw conclusive evidence about how those outcomes related to specific teaching practices. Others provided detailed explanations of pedagogical practice yet made unsubstantiated claims about, or provided only inferential evidence for, how those practices connected with student outcomes.

Granted, we were not looking for linear explanations. As Sfard (2005) points out, the complexity of the teaching–learning relationship “precludes the possibility of identifying clear-cut cause–effect relationships” (p. 407). What we were searching for were studies that were able “to offer a developing picture of what it looks like for a teacher’s practice to cultivate student [proficiency]” (Blanton & Kaput, 2005, p. 440). We were searching for studies that offered a “detailed look at how [teachers’] actions played out in the classroom and how students were involved in this” (Blanton & Kaput, 2005, p. 435) and the sorts of mathematical proficiency that resulted. Specifically, we were seeking studies that offered not just detailed descriptions of pedagogy and outcomes but rigorous explanation for close associations between pedagogical practice and particular outcomes.

Paying attention to diverse forms of research evidence required our serious consideration of the literature relating to disparate factors from different sectors and representative of different time periods. Luke and Hogan (in press) note that what is distinctive about the approach undertaken in the New Zealand Best Evidence programme “is its willingness to consider all forms of research evidence regardless of methodological paradigms and ideological rectitude, and its concern in finding contextually effective appropriate and locally powerful examples of ‘what works’... with particular populations, in particular settings, to particular educational ends” (p. 5). We have included many different kinds of evidence that take into account human volition, programme variability, cultural diversity, and multiple perspectives. Each form of evidence, characterised by its own way of looking at the world, has led to different kinds of truth claims and different ways of investigating the truth. Our pluralist stance left us free to consider the relative strengths and weaknesses of different methodological approaches.

A fundamental challenge for this BES has been to demonstrate a basis for knowledge claims. We are absolutely aware that, like data selection, assessment of evidential claims from secondary sources is a highly perspectival activity. “Even those gazing down a microscope are as capable of finding what they expect to find, or want to find, as anyone else” (Davies, 2003). In response to this challenge, studies have been reported in a way that will make the original evidence as transparent as possible. Informed by the *Guidelines for Generating a Best Evidence Synthesis Iteration 2004*, we included studies that:

- provided a description of the context, the sample, and the data;

- offered details about the particular pedagogy and the specific outcomes;
- connected research to relevant literature and theory;
- used methods that allow investigation of the pedagogy–outcome link;
- yielded findings that illuminated what did or did not work.

The *Guidelines for Generating a Best Evidence Synthesis Iteration* allowed us to deal not only with a diversity of research topics, approaches, and methods, but also to capture differences in the context, practices, and ways of thinking of researchers. The method employed in this BES for evaluating validity required us to look at the ways different pieces of data meshed together and to determine the plausibility, coherence, and trustworthiness of the interpretation offered.

Assessments about the quality of research depend to a large extent on the nature of the knowledge claims made and the degree of explanatory coherence between those claims and the evidence provided. What we were looking for was the explanatory power of the stated pedagogy–outcome link. When assessing the nature and strength of the causal relations between pedagogical approaches and learning outcomes, we were guided by Maxwell’s (2004) categorisations of two types of explanations of causality. The first type, the *regularity view of causation*, is based on observed regularities across a number of cases. The second type, *process-oriented explanations*, sees “causality as fundamentally referring to the actual causal mechanisms and processes that are involved in particular events and situations” (p. 4). Cobb argues (2006, personal communication) that regularity explanations are particularly useful for policy makers, while process-oriented explanations are relevant to teachers, who are concerned with “the mechanism through which and the conditions under which that causal relationship holds” (Shadish, Cook, & Campbell, 2002, p. 9, cited in Maxwell, 2004, p. 4). Attending to both types of explanation of causality meant including both large-scale and single-case studies. In many instances, we have found it useful to present a single case—a learner or teacher, a classroom, or a school—in the form of a vignette to exemplify the relations between learning processes and the means by which they are supported.

Research sources in this BES report

This BES report contains approximately 660 references. Included amongst these are research reports of empirical studies, ranging from very small, single-site settings (e.g., Hunter, 2002) to large-scale longitudinal studies (e.g., Balfanz, MacIver, and Byrnes, 2006). Some of the larger studies have multiple references because they include different papers/conference proceedings/book chapters or because they embrace work authored by different researchers (e.g., the New Zealand Numeracy Development Project). In addition, the references include reports containing educational statistics and policy, theoretical writings, and commentaries and reviews on multiple research findings (e.g., van Tassel-Baska, 1997) .

The *Guidelines for Generating a Best Evidence Synthesis Iteration* point to the importance of drawing on New Zealand research in order to illuminate what works in the New Zealand context. However, despite an exhaustive search for New Zealand work, it is apparent (see chapter 8 for further discussion) that the strengths and foci of New Zealand research are not evenly distributed. In some areas—for example, early years education—there are relatively few New Zealand (or Australian) researchers working with a specific focus on mathematics education (Walshaw & Anthony, 2004). Table 2 shows the country of origin of the literature included in this BES. The numbers reflect New Zealand’s relatively new positioning within the international mathematics education research community.

Table 2: Database composition according to country

Country	Relative Frequency
New Zealand	27%
Australia	17%
United Kingdom	11%
United States	49%
Other (e.g., Africa, Netherlands, Spain)	6%

Table 3 shows the proportion of the items included in the BES (both empirical studies and commentaries) that relates to each of the different sectors. Publications relating specifically to intermediate schools have been classified with the literature on primary schools.

Table 3: Database composition according to sector

Sector	Relative Frequency (n=520)
Preschool	18%
Primary school	48%
Secondary school	21%
Teacher education	13%

Synthesising the data

Our conceptual framework, outlined in chapter 2, offered a way of structuring the data. Within the community of practice frame in and beyond the classroom, we identified the following components: (a) the organisation of activities and the associated norms of participation, (b) discourse, particularly norms of mathematical argumentation, (c) the instructional tasks, and (d) the tools and resources that learners use. We began the iterative chapter-structuring process by outlining a number of key areas. These included mathematical thinking and identities, scaffolding and co-construction, tasks, activities, assessment, educational leadership, home-school/centre links, and wider school communities. Each of these served as a starting point for our exploration and was found, in the course of the investigation, to be a useful initial category for addressing questions of equity and proficiency in relation to effective mathematics teaching.

In time, we organised these categories more cohesively into groups. What we endeavoured to do was organise multiple elements, types, and levels and varying temporal conditions in line with the critical dimensions of a community of practice and the guiding principles established in chapter 2. The content of the subsequent chapters is shaped according to these dimensions and principles. Chapter 3 focuses on all three dimensions in a search for understanding of how pedagogy influences early years outcomes. Chapters 4 and 6 explore interrelationships that are centred on the joint enterprise of developing mathematical proficiency for all learners. Chapter 5 explores the role of mathematical tasks and the part that they play in enhancing students' learning.

Reminding ourselves and readers that this BES synthesis is a product of currently accessible research, we concur with Atkinson's (2000) view that "the purpose of educational research is surely not merely to provide 'answers' to the problems of the next decade or so, but to continue to inform discussion, among practitioners, researchers and policy-makers, about the nature, purpose and content of the educational enterprise" (p. 328). Rather than offering broad answers that promise much and achieve little, it is our hope that the structure we have used will foster understanding, reflection, and action concerning the characteristics of effective pedagogical approaches in mathematics.

References

- Atkinson, E. (2000). In defence of ideas, of why 'what works' is not enough. *British Journal of Sociology of Education*, 21(3), 316-330.
- Balfanz, R., Maclver, D., & Brynes, V. (2006). The implementation and impact of evidence-based mathematics reforms in high-poverty middle schools: A multi-site, multi-year study. *Journal for Research in Mathematics Education*, 37(1), 33-64.
- Bishop, A., Clements, K., Keitel, C., Kilpatrick, J., & Leung, D. (Eds.). (2003). *Second international handbook of mathematics education*. Dordrecht: Kluwer.
- Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36(5), 412-446.
- Davies, B. (2003). Death to critique and dissent? The politics and practices of new managerialism and of 'evidence-based practice'. *Gender and Education*, 15(1), 91-103.
- Luke, A., & Hogan, D. (2006). Redesigning what counts as evidence in educational policy: The Singapore model. In J. Ozga, T. Popkewitz, & T. Seddon (Eds.), *Education research and policy: Steering the knowledge-based economy. World handbook of education* (pp. 170-184). London: Routledge.
- Maxwell, J. (2004). Causal explanation, qualitative research, and scientific inquiry in education. *Educational Researcher*, 33(2), 3-11.
- Ministry of Education. (2004). *Guidelines for generating a Best Evidence Synthesis Iteration 2004*. Wellington: Ministry of Education.
- Hunter, R. (2002). *Constructing decimal concepts in an inquiry classroom*. Unpublished Master's thesis, Massey University, Palmerston North, New Zealand.
- Sfard, A. (2005). What could be more practical than good research? On mutual relations between research and practice of mathematics education. *Educational Studies in Mathematics*, 58, 393-413.
- Walshaw, M., & Anthony, G. (2004). Research trends in mathematics education. In B. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 3-28). Sydney: Post Pressed.
- van Tassel-Baska, J. (1997). What matters in curriculum for gifted learners: Reflections on theory, research and practice. In N. Colangelo & G. Davis (Eds.), *Handbook of gifted education* (pp. 126-135). Boston, MA: Allyn and Bacon.

Appendix 2: URLs of citations

The following 22 papers/articles/chapters/books are suggested as potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration. Readers are encouraged to source and read them. Several are available online; the others can be sourced through libraries.

The full citations are hyperlinked in the online PDF. For the convenience of those using a hard copy of the text, the URLs are listed below.

Carpenter, Thomas P ; Franke, Megan L ; Jacobs, Victoria R

A longitudinal study of invention and understanding in children's multidigit addition and subtraction
<http://nzcer.org.nz/BES.php?id=BES001>

Clarke, Barbara ; Clarke, Doug

Mathematics teaching in Grades K-2: painting a picture of challenging supportive, and effective classrooms
<http://nzcer.org.nz/BES.php?id=BES002>

Cobb, Paul ; Boufi, Ada ; McClain, Kay ; Whitenack, Jor

Reflective discourse and collective reflection
<http://nzcer.org.nz/BES.php?id=BES020>

Empson, Susan B

Low performing students and teaching fractions for understanding: An interactions analysis
<http://nzcer.org.nz/BES.php?id=BES021>

Fraivillig, Judith L ; Murphy, Laren A ; Fuson, Karen C

Advancing children's mathematical thinking in everyday mathematics classrooms
<http://nzcer.org.nz/BES.php?id=BES003>

Gifford, Sue

A new mathematics pedagogy for the early years: in search of principles for practice
<http://nzcer.org.nz/BES.php?id=BES004>

Goos, Marilyn

Learning mathematics a classroom community of inquiry
<http://nzcer.org.nz/BES.php?id=BES005>

Houssart, Jenny

Simplification and repetition of mathematical tasks: a recipe for success or failure?
<http://nzcer.org.nz/BES.php?id=BES006>

Irwin, Kathie ; Woodward, J (paper available online)

A snapshot of the discourse used in mathematics where students are mostly Pasifika (a case study in two classrooms)
<http://nzcer.org.nz/BES.php?id=BES007>

Kazemi, Elham ; Stipek, Deborah

Promoting conceptual thinking in four upper-elementary mathematics classrooms
<http://nzcer.org.nz/BES.php?id=BES008>

Latu, Viliami (paper available online)

Language factors that affect mathematics teaching and learning of Pasifika students
<http://nzcer.org.nz/BES.php?id=BES009>

O'Connor, Mary Catherine

"Can any fraction be turned into a decimal?" A case study of the mathematical group discussion
<http://nzcer.org.nz/BES.php?id=BES010>

Rietveld, Christine M.

Classroom learning experiences of mathematics by new entrant children with Down syndrome
<http://nzcer.org.nz/BES.php?id=BES011>

Savell, Jan ; Anthony, Glenda Joy

Crossing the home-school boundary in mathematics
<http://nzcer.org.nz/BES.php?id=BES049>

Sheldon, Steven B ; Epstein, Joyce L

Involvement counts: family and community partnerships and mathematics achievement

<http://nzcer.org.nz/BES.php?id=BESo12>

Smith, Margaret Schwan Smith ; Henningsen, Marjorie A

Implementing standards-based mathematics instruction: a casebook for professional development

<http://nzcer.org.nz/BES.php?id=BESo13>

Steinberg, Ruth M ; Empson, Susan B ; Carpenter, Thomas P

Inquiry into children's mathematical thinking as a means to teacher change

<http://nzcer.org.nz/BES.php?id=BESo14>

Watson, Anne ; De Geest, Els

Principled teaching for deep progress: Improving mathematical learning beyond methods and material

<http://nzcer.org.nz/BES.php?id=BESo15>

Wood, Terry (paper available online)

What does it mean to teach mathematics differently?

<http://nzcer.org.nz/BES.php?id=BESo16>

Yackel, Erna ; Cobb, Paul

Sociomathematical norms, argumentation, and autonomy in mathematics

<http://nzcer.org.nz/BES.php?id=BESo17>

Young-Loveridge, Jenny (paper available online)

Students views about mathematics learning: a case study of one school involved in Great Expectations Project

<http://nzcer.org.nz/BES.php?id=BESo18>

Zevenbergen, R

The construction of a mathematical habitus: implications of ability grouping in the middle years

<http://nzcer.org.nz/BES.php?id=BESo19>

Appendix 3: Glossary

The page reference for the first and/or most important occurrence of the term is given in brackets.

- Cognitive engagement** (p. 2). The state of being engaged in thinking
- Community of Practice** (p. 6). The complex network of relationships within which teachers teach and students learn
- Connectionist teachers** (p. 97). Teachers who consistently make connections between different aspects of mathematics
- Decile** (p. 9). In New Zealand, a 1–10 system used by the Ministry of Education to indicate the socio-economic status of the communities from which schools draw their students; low-decile schools receive a higher level of government funding
- Developmental progressions** (p. 47). Sequential learning pathways categorised as a series of steps
- Empirical evidence** (p. 24). Data that has been collected systematically for research purposes
- Equity** (p. 9). The principle based on the belief that social injustices should be redressed by allocating resources according to need, not power; in education, this may mean, amongst other things, the provision of different pedagogical approaches depending upon the needs of the learners
- Family Math** (p. 171). A US initiative designed to develop parents' skills so they can work with their children on their mathematics
- Feed the Mind** (p. 45). A media campaign funded by the New Zealand Ministry of Education and designed to support family involvement in children's learning
- High or low press for understanding** (p. 121). Differing levels of cognitive engagement demanded of students by teachers for clarification of thinking
- Kahoa** (p. 36). A festive necklace (Tongan)
- Kōhanga reo** (p. 9). Māori-medium early childhood centres
- Kura kaupapa Māori** (p. 10). Māori-medium schools (kura = school), based on a Māori philosophy of learning (see pp. 54–5)
- Manipulatives** (p. 133). Any concrete materials used by students to model mathematical relationships
- Mathematical argumentation** (p. 123). Presenting a case to support or refute a premise developed by mathematical thinking
- Mathematical identity** (p. 19). How a student sees him/herself as a learner and doer of mathematics
- Metacognition** (p. 38). The knowledge and processes involved in thinking about and regulating one's own thinking, which is essential for reflecting, self-monitoring, and planning
- Norms of participation** (p. 54). The rules, spoken or unspoken, that govern the way students behave and contribute in the classroom
- Number Framework** (p. 109). A model, structured in 8 stages, showing how students typically develop understanding of number and number operations (New Zealand, NDP)
- Number sense** (p. 98). An understanding of the relationships, patterns, and fundamental reasonableness that lie behind all mathematical operations
- Numeracy** (p. 28). The ability to use mathematics effectively, fluently, and with understanding in a wide variety of contexts
- Numeracy Development Project (NDP)** (pp. 9, 17). The central part of the New Zealand Ministry of Education's Numeracy Strategy, which has as its primary objective the raising of student achievement in numeracy through lifting teachers' professional capability
- NumPA** (p. 9). A structured, diagnostic interview used by teachers to place students on the early stages of the Number Framework (New Zealand, NDP)
- Open-ended tasks** (p. 106). Tasks that require students to engage in problem definition and formulation before beginning to think about a solution
- Pasifika students** (p. 9). Students whose families have come from Sāmoa, Tonga, the Cook Islands, Niue, Tokelau, Tuvalu, and some other, smaller Pacific nations
- Pedagogical Content Knowledge** (p. 199). In this context, knowledge about mathematics and how to teach it as well as knowledge about how to understand students' thinking about mathematics
- Pedagogy** (p. 5). The processes and actions by which teachers engage students in learning
- Poi** (p. 26). A small ball, often made of woven flax, on the end of a length of string; swung rhythmically by women when performing action songs (Māori)
- QUASAR** (p. 95). A programme developed to help urban students develop understanding of mathematical ideas through engagement with challenging mathematical tasks
- Revoicing** (p. 78). The repeating, rephrasing, or expansion of student talk in order to clarify or highlight content, extend reasoning, introduce new ideas, or move discussion in another direction
- Scaffolding** (p. 27). Temporary, structured support designed to move learners forward in their thinking

- School–home or home–school partnership** (p. 160). The deliberate nurturing of relationships between the school and the home, in the interests of better supporting student learning
- Sociocultural practices** (p. 19). Practices relating to the social and cultural aspects of participation in the classroom
- Sociocultural theory** (p. 24). The theory that learning arises out of social interaction
- Socio-economic status (SES)** (p. 30). Categorisation of individuals or communities, based on income, family background, and qualifications
- Sociomathematical norms** (pp. 61–62). Shared understandings of the processes by which students and teacher contribute to a mathematical discussion
- Tasks** (p. 94). Defined by Doyle (1983) as “products that students are expected to produce, the operations that students are expected to use to generate those products, and the resources available to students while they are generating the products”
- Te ao Māori** (p. 54). The Māori world
- Te Poutama Tau** (p. 59). The Numeracy Project (New Zealand) as developed for implementation in Māori-medium schools
- Te Whāriki** (p. 24). The New Zealand early childhood curriculum (for children aged 5 or under)
- Tukutuku panels** (p. 115). A Māori craft form consisting of ornamental lattice-work panels woven together with strips of flax into intricate designs
- Waiata** (p. 26). A song (Māori)
- Whānau** (p. 41). Extended family (Māori)
- Wharekura** (p. 9). Māori-medium secondary schools, which are based on a Māori philosophy of learning
- Zone of Proximal Development (ZPD)** (p. 36). Vygotsky (1986) describes the ZPD as the “distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers”

Abbreviations

CGI	Cognitively Guided Instruction Project	pp. 17, 105
EAL	English as an Additional Language	p. 116
EFTPOS	Electronic Funds Transfer at Point of Sale	p. 115
EMI-4s	Enhancing the Mathematics of Four-Year-Olds.	p. 28
ENRP	Early Numeracy Research Project	p. 158
EPPE	Effective Provision of Pre-school Education Project.	p. 25
ERO	Education Review Office.	p. 158
IAMP	Improving Attainment in Mathematics Project	pp. 18, 99
ICME	International Congress on Mathematics Education	p. 20
ICT	Information and Communication Technologies	p. 27
IEA	International Association for the Evaluation of Educational Achievement	p. 154
IMPACT	Increasing the Mathematical Power of All Children and Teachers	p. 73
MEP	Mathematics Enhancement Project.	p. 60
NCEA	National Certificate of Educational Achievement	pp. 10, 66
NEMP	National Education Monitoring Project	p. 9
NNS	National Numeracy Strategy	p. 17
PISA	Program for International Student Assessment.	p. 8
REPEY	Researching Effective Pedagogy in the Early Years	p. 25
RME	Realistic Mathematics Education	p. 113
TIMSS	Third International Mathematics and Science Study.	p. 14
VAMP	Values and Mathematics Project	p. 58

Index

A

- algebra 98–99, 126, 139–140
- artefacts *see* tools
- assessment 152
 - in early years 33–35, 47
 - informal 112
- Australia
 - classroom interaction 15–16
 - early childhood centre programmes 31, 44
 - early childhood computer use 27–28
 - early numeracy projects 16, 47–48, 158–159
 - families and numeracy 41, 44
 - language differences 69
 - maths as challenging experience 120
 - Overcoming Barriers to Learning Project 20
 - problem-solving among able and less able students 118–119
 - professional development 158
 - technology and students' attitudes 139
 - in TIMSS Video Study 14, 15

B

- Berkeley University 171
- Best Evidence Synthesis (BES) xviii, 1, 14, 41, 202, 206–209
- biculturalism 6, 9, 202
- bilingual students *see* language
- books *see* reading and numeracy
- Britain *see* England; United Kingdom

C

- calculators 136–137, 139–140
- calculus 106, 129, 137–138
- Canada
 - pocket-money study 169
 - pre-schoolers 43–44
- caring about students 2, 54, 56, 86, 203
- CGI (Cognitively Guided Instruction) *see* United States
- Checkout/Rapua* 47
- Chinese students
 - in New Zealand 70
 - in United States 160–161
- Chinese teachers 197–198
- classroom 5
 - see also* caring about students; feedback; group work; interaction; language
 - daily practices 61–62
 - discussion 2, 60–61, 63, 72–76, 79–81, 124, 131, 178, 193, 203
 - inequities 57, 74
 - promotion of risk-taking 55
 - social context 18

- cognitive development 55, 65–66, 79–81, 118, 120–124
 - pre-schoolers 169
- community involvement 3–4, 21, 160, 165–167, 171, 202
 - see also* families/whānau; parents
- computer use *see* information and communications technologies (ICT)
- concepts (mathematical) 3, 15, 69–70
 - measurement 30–31
- connectionist teachers 97, 102
- contexts *see* tasks
- counting and number challenged as entry point 31
- cultural context of learning 46, 115, 119
- cultural differences/backgrounds 54
- culture, role of 157
- curriculum xx, 95, 115, 122–123, 151, 170
 - community involvement 44, 165–166
 - differentiation 64, 120
 - pre- and early school years 30–31, 46–48; *see also* *Te Whāriki*
 - reforms and changes 4, 10, 129
 - school leaders and 152–155
 - teachers' knowledge of 46, 82–83, 158
- Curriculum Framework, the New Zealand xix, 7
- Cyprus
 - parents in classroom 167
- Czech Republic
 - in TIMSS Video Study (mentioned) 14

D

- Davydov, V.V. 21
- decimals 116
- deficit theories 8, 60
- department heads *see* lead teachers
- disabilities, students with 60, 64, 119, 133, 136
 - see also* special needs teaching
- disadvantaged students 60, 110, 118, 152
 - see also* social inequity; socio-economic status

E

- early childhood education 24–48
 - see also* games and numeracy; play as means of learning; pre-schoolers
 - assessment 33, 47
 - centres 4, 202, 204
 - teacher knowledge 2, 45–46
 - computer/ICT use 27–28
 - curriculum 46–48
 - fantasy contexts 113
 - group work 25, 33
 - measurement activities 31
 - scaffolding 36–37, 41
 - social interactions 38–39, 46
 - work-play balance 25

- Education, Ministry of *see* Ministry of Education
- Education Review Office (ERO) 152
- Effective Numeracy Teaching (Pasifika Focus) project 60
- England
see also United Kingdom
- deaf children, programme for 136
- National Numeracy Strategy/Framework 17, 111
- pre-schoolers, study of 43
- streamed classes and sets 57, 118
- Enhancing the Mathematics of Four-Year-Olds (EMI-4s) project 30, 32, 35
- equity 8–10, 18
see also socio-economic status
- ethnic group comparisons 9, 18, 42, 83, 170
- expectations for students 56
- F**
- families/whānau 3, 10
see also home-school links; parents
- home activities and practices 41–44, 46, 162, 169
- support for learning 20, 178–179
- Feed the Mind campaign 45
- feedback 76–78, 124–125
- fractions ix, 54, 75, 126, 177–199
- G**
- games and numeracy 30, 32, 120, 165, 169, 170 171, 178–179
see also play as means of learning
- gender differences 56, 139
- geometry 108–109, 133, 138
- gifted students *see* high-achieving students
- goals of mathematics teaching 6–7, 18–19
- group work 2, 64–68, 86, 190, 203
 early childhood 25, 33
 multilingual classrooms 70–71
 primary school 67, 108
- H**
- Hawaii
 study of money activities and knowledge 169
- high-achieving students 64, 66, 98, 119–120
- home-school links 44–45, 151, 160–172, 202
 communication 164–165, 167
- homework 167–168
- Hong Kong
 lessons for silent learners 15
 in TIMSS Video Study (mentioned) 14
- I**
- IAMP *see under* United Kingdom
- IEA (International Association for the Evaluation of Education Achievement) study 155, 168
- immigrant families (in UK) 168
- immigrant students 60, 63
- IMPACT project 73, 171
- individual/independent learning 68–69
- informal learning 25–27
- information and communications technologies (ICT) 136–140
 in early childhood education 27–28
 for special needs child 125
- inscriptions 126–127
- interaction of teacher and class 15–16, 55, 63, 75, 79–81, 104, 112, 119
see also feedback
- International Congress on Mathematics Education (ICME) 20
- international research 14–21
- Internet and email 139
- intuitive techniques of young children 35
- J**
- Japan
 addition as challenging experience 120
 discussion and whole-class solutions 15
 in TIMSS Video Study 14
 tools to communicate thinking 136
 use of students' solutions 102–104
 verbal explanations of concepts 72
- Johns Hopkins University 160
- K**
- Kei Tua o te Pae/Assessment for Learning* (document) 33, 39, 44
- kinaesthetic learners 15, 132
- knowledge, prior or informal (children's) 30, 113–116, 203
- kura kaupapa Māori 54–55, 165
- L**
- language 71–72, 203
 mathematical 2, 69–72, 78, 86, 204; *see also* vocabulary/word problems
- multilingual classrooms 2, 70–71, 204
- lead teachers (department heads) 154–155, 157, 158
- leadership teams 152–155, 158, 159
- Literacy and Numeracy Strategy (NZ) 159
- low-achieving students
 in classroom discussions 63, 65–66
 helped by greater challenges 18, 119, 121, 183
 helped by problem-solving tasks 95, 118, 179–183
 perception of mathematical structure 98, 99–100
 use of manipulatives 133
- low-decile schools 9, 158
- M**
- Māori children 9, 56
- Māori community involvement 165–166
- Māori culture and language 54, 119

- Māori families 161
- Māori-medium teaching 9, 11, 59–60, 157–158
- Māori students 60, 73, 114, 119, 132
- mathematical understanding in young people 24, 33, 40
- mathematicians' practices, development of 18–19
- Mathematics Enhancement Project (NZ) 158
- Mathematics in the New Zealand Curriculum* 49
- Mauritius
- study of home-related factors 161
- metacognition 38–40, 108, 177
- Ministry of Education xviii, 1, 45, 164, 205
- Numeracy Strategy 59, 159;
 - see also* Numeracy Development Project (NDP)
- modelling activities 106–108, 133–135
- money and numeracy 115, 169–170
- motivation (among students) 162
- multiculturalism 9, 170
- N**
- National Council of Teachers of Mathematics (NCTM) 6
- National Education Monitoring Project 9, 114
- National Research Council 7
- NCEA (National Certificate of Educational Achievement) 10, 124, 139, 152
- Netherlands
- contextually based tasks 113–114
 - use of equipment 132
- newsletters (for families) 162, 164
- Numeracy Development Project (NDP) 9, 16, 17, 18, 45, 111, 131, 177
- classroom language 71
 - development of teachers 83, 111, 159
 - equipment, use of 129, 132
 - group and peer work 63, 65
 - Number Framework 109, 113, 157, 167
 - problem-based tasks 95, 98
- numeracy skills, stability of 47–48
- O**
- OECD (Organisation for Economic Co-operation and Development) reports 8, 161
- P**
- parents 160–171
- see also* families/whānau; home-school links
 - in classroom 167
 - communication with 44, 164–165
 - involvement of 30, 162–169, 171, 202
 - language skills 164
 - parental education programmes 45, 171
 - parental hesitancy or alienation 161, 164–165, 171
- Pasifika
- see also* Tongan students and families
 - families 161
 - students 9, 60, 70–71, 73, 74, 132, 204
- Pasifika Focus 60
- path smoothing 121
- patterning skills 31, 36
- pedagogical approaches 5–6, 199
- pedagogical practice 10, 17, 18–20, 63, 70, 139, 184, 202
- peer interaction (students') 65–68, 97, 126
- see also* group work
- piloting 82
- PISA (Programme for International Student Assessment) 8, 9, 160
- play as means of learning 26–28, 45, 46, 203
- see also* games and numeracy
- pre-schoolers
- see also* early childhood education
 - family's role 31, 41–46, 162, 169
 - mathematical thinking of 24, 30, 31, 40, 48
- principals, role of 152, 158, 165
- professional development 17, 111, 158–159
- early childhood teachers 46
 - early years educators 49
 - Māori-medium teachers 59
 - school leadership and 153
- Q**
- qualification levels 9
- R**
- Radzikhovskii, L.A. 21
- reading and numeracy 43–44, 170–171
- Realistic Mathematics Education (RME) programme 113–114, 132
- reforms 10, 16, 17, 152, 154, 158, 159
- relationship of teacher and students 56–61, 131
- see also* interaction
- research 204, 206–209
- gaps in 4, 204–205
 - international 14–21
- resource persons 156
- resources *see* tools
- revoicing 78–79
- S**
- scaffolding 36–37, 72, 121, 122, 123
- examples 76, 125, 197
- School Entry Assessment 47
- school-wide approaches 151–152, 159–160
- see also* whole-school partnerships
- social and economic context of mathematics 5, 78
- social inequity 8–10
- see also* socio-economic status
- social nurturing 58–60, 160
- social perspective of learners 35, 36
- social skills and relationships 65, 67–68, 96–97
- socio-economic status 40, 42, 46, 74, 98, 114, 115, 161

- South Africa
 mathematical language vs reasoning 71
 use of socio-cultural context 115
- Spain
 immigrant students in 63
 special needs teaching 10, 119, 125–126
see also disabilities, students with
 statistics 4, 10, 127, 130
 streamed classes and sets 2, 57, 58, 64, 118–119, 203
 student diversity 8–9, 60, 162
 student engagement 15, 56, 62–64, 104–105, 119, 203
 exchange of ideas 72–76, 79–81
 learning logs 131
 students' mathematical thinking 78–81, 85, 102
 104, 112, 127–129, 188
 pre-schoolers 40
 teachers' engagement with 157, 195–197
 support of colleagues 154–156, 159
see also whole-school partnerships
- T**
 tasks 94, 204
 cognitive development through 120–124
 completion as goal, flaws of 122, 183
 contextually-based 113–117
 learners' existing knowledge and 111–113, 116
 open-ended 106–109, 110, 115–116
 problem-solving 94–108, 113, 122–123, 179
 183, 191
 task differentiation 110
 worked examples 104–106
- te ao Māori (the Māori world) 54
- Te Poutama Tau 11, 59–60, 109, 157–158, 167, 177
- Te Whāriki* (New Zealand early childhood curriculum) 7, 21, 24, 47, 49
- teacher effectiveness 5, 10
- teacher knowledge 17, 81–85, 110–111, 197–199, 203–204
see also professional development
 early childhood 45–46
- teachers' attitudes 56–58, 63, 74
- teachers' practice, traditional and non-traditional 55
- technological tools 136–140
 students' attitudes to 139
- textbooks 131–132
- time (for teaching) 155
- TIMSS (Third International Mathematics and Science Study) Video Study 14–15
- Tongan students and families 70, 168
- tools 126–136, 187
- Treaty relationship 9
- U**
 United Kingdom
see also England
 assessment and testing 112, 114
 Effective Teachers of Numeracy project 16, 82–83
 family numeracy programmes 44
 feedback evaluated 77
 IAMP (Improving Attainment in Mathematics Project) 18–19, 99, 114, 124, 125
 immigrant families 168
 IMPACT (Increasing the Mathematical Power of All Children and Teachers) project 73, 171
 lead mathematics teachers 154
 low-attaining students 121, 183–184
 modelling in Welsh schools 108
 parental involvement 165
 pre-school/early childhood teaching 25, 45
 REPEY research project 25, 31–32, 45
 special needs teaching 125
 streamed classes 64
 textbooks 132
- United States 18, 72, 133
 Chinese students in 160–161
 Cognitively Guided Instruction (CGI) project 17, 111, 156–157
 Family Math programme 171
 Focus on Results in Math study 20
 low-income families and districts 30, 152, 153, 158, 161
 pre-schoolers 28, 42
 Big Math for Little Kids programme 40
 Head Start programme 30–31
 Number Worlds programme 30
 problem-solving approaches 95–96
 professional development 46, 111
 QUASAR middle-school programme 95, 118, 121
 Talent Development (TD) Middle School programme 95
 teacher knowledge 197–198
 in TIMSS Video Study (mentioned) 14, 15
 university entrance 9
- V**
 vocabulary/word problems 116–117, 164
 Vygotsky, L. 21, 36
- W**
 Wales: modelling in schools 108
 whole-school partnerships 157–160
see also school-wide approaches